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Wave behavior in anharmonic Penrose lattices

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Taking advantage of the close resemblance of Kirchhoff's laws and the equations of motion of isotropic phonons in a simply connected lattice, a device to directly observe the vibrational eigen-modes of a Penrose lattice was built with electric LC oscillators. The wave interference patterns observed in this network reveal key aspects concerning the localization of states, which has not been settled so far. There are truly extended eigen-states at low frequencies, and critically localized ones when their wavelength is of the order of the distance between neighbors in the lattice. The device allows the study of non-linear effects, when it is excited with higher voltages. These effects are analyzed by perturbation theory and the observed shifts of the eigen-frequencies are explained as a consequence of adding anharmonic terms to the Hamiltonian.

1. Introduction

Quasiperiodicity and non-linear dynamics have been of tremendous interest in recent years. One of the most controversial aspects has been the influence of quasiperiodicity in the localization of the wave functions. Theoretical studies predict a critical spectrum in one [1] and two dimensions [2], and one could expect peculiar transport properties from such a band structure [3]. Transport experiments in real quasicrystals are not very illuminating so far [4], however, extremely interesting results have been obtained from artificial quasicrystals, as Fibonacci superlattices [5], where there is evidence of Brillouin zone folding for the acoustic bands [6]. There have also been two-dimensional analog simulations, as a Penrose array of Josephson junctions [7], or an array of tuning forks at the centers of the Penrose rhombi, to study acoustic interference [8].

In this paper we report on an analog simulation of wave interference in an array of oscillators disposed as in the vertex problem in the Penrose lattice. Such a simulation consists of LC oscillators, forming a T-line [9], that is, each vertex contains a grounded capacitor and the bonds are equal inductances (see fig. 1). This experiment has the advantage that it is simple, illustrative and does not require sophisticated equipment to measure. The other advantage is that the results for small oscillations (small voltages) can be calculated theoretically in exact form. The assumption of small amplitudes is not necessarily met in the real simulator, and this makes it an ideal device to study anharmonic and non-linear effects in the Penrose tiles.

2. Experiment

An LC circuit is analogous to a system of masses linked by springs, the normal modes at a given frequency (ω) are determined by Kirchhoff's laws as the solutions of the equation

$$\mathrm{i}\omega C V_n + \frac{1}{\mathrm{i}\omega L}\sum_m (V_n - V_m) = 0, \tag{1}$$

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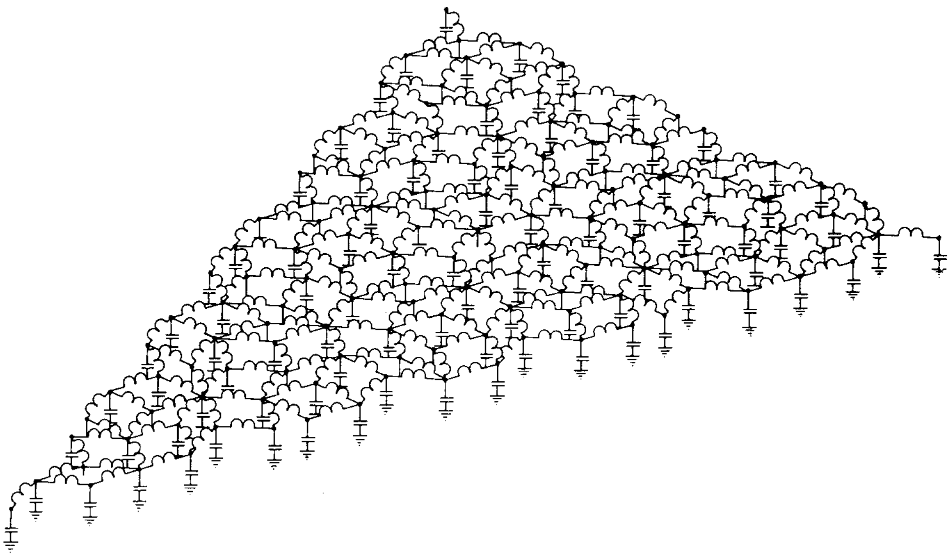

Fig. 1. Sketch of the network of LC oscillators, connected as in the vertex problem of a Penrose lattice. The bonds between neighbors are inductances and there is a grounded capacitor at each vertex.

where the inductance (L) and the capacitance (C) are the same in all sites, and V_n is the voltage measured in site n. Here the summation spans all sites m directly connected to site n. Equation (1) has the same eigenvalues of the secular equation for phonons [10]

$$-M\omega^2 + K\sum_m (U_n - U_m) = 0, \tag{2}$$

where the U_n are the displacements of masses M, connected by isotropic springs with a force constant K. This analogy allows one to transfer the results obtained with this circuit, to the problem of excitations of the tight-binding type Hamiltonians in the quasicrystal. The boundary conditions for eq. (1) depend on the way one feeds the system; in our case a constant current was induced in site $n = 1$, which is situated at the most acute angle of the triangle of fig. 1. Therefore, the first of eqs. (1) has an inhomogeneous term $I(\omega)$.

The actual apparatus corresponds to a tile of generation 13 of the Penrose lattice, defined according to the inflation rule described elsewhere [11]. This array contains 137 vertices and 240 bonds. The value of the capacitances in the vertices was 1 μF ± 10%, and the inductances of 7.8 mH ± 5% were made with toroids of ferrite with a relative magnetic permeability of $\mu/\mu_0 \cong 1500$. The choice of the toroidal shape was due to the need of minimizing the problem of mutual inductances. The dissipative effects of the inductances were examined experimentally and simulated in the theory by an imaginary term $-R(\omega, \Delta V)/\omega$, added to L. At a peak-to-peak voltage of $\Delta V = 0.2$

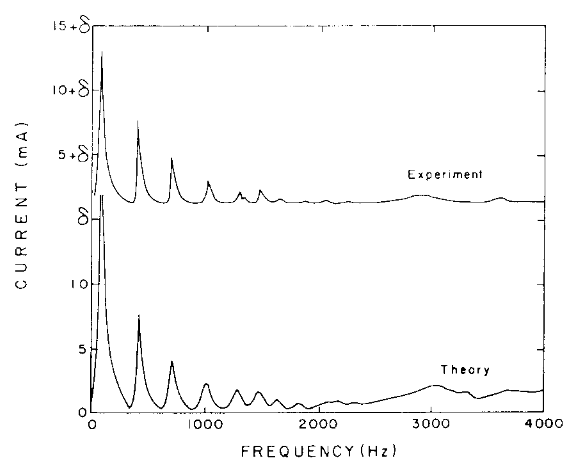

Fig. 2. Experimental spectrum obtained from the device of fig. 1, when a current is induced through the site at the acute extreme of the triangle of fig. 1. The theoretical spectrum from eq. (1) is also shown for comparison. The vertical scale has been shifted by δ.

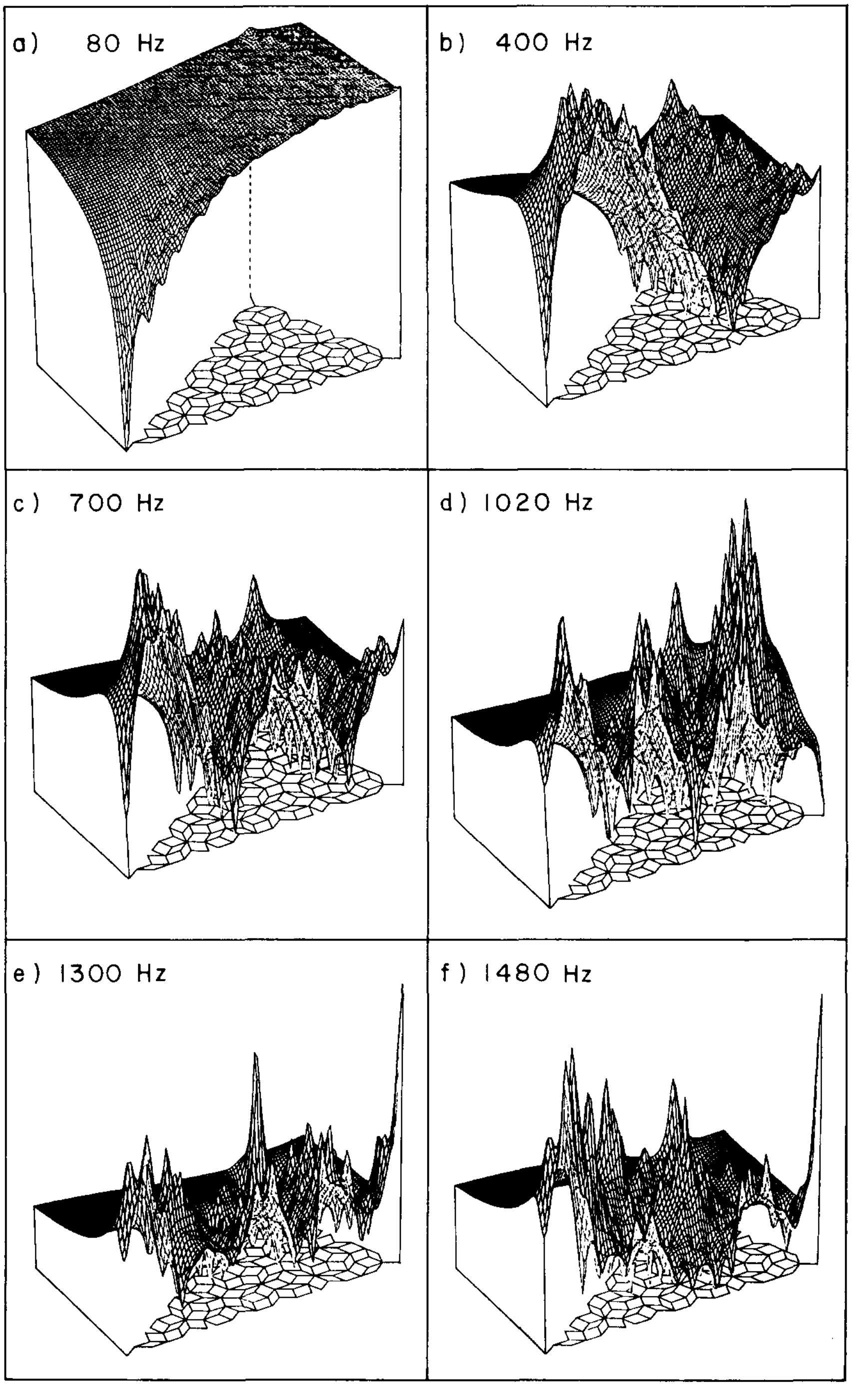

Fig. 3. Spatial distribution of the site amplitudes for the first six low-frequency eigen-states of fig. 2. The respective eigenvalues are indicated in each case, and the vertical height represents the measured voltages.

V, $R(\omega) = 0.006\omega^{0.7}$, and at fixed frequency (ω = 500 Hz), $R(\Delta V) = 5.6\Delta V$. On the other hand, the real part of L is also a function of the applied voltage, for instance, at ω = 500 Hz, $L(\Delta V) \cong 7.7 + 4.5\Delta V$.

The experiment is done by feeding a monochromatic current to one extreme of the lattice at a peak-to-peak voltage of 0.2 V. The measurement of this current as a function of frequency is shown in fig. 2, where we compare the predicted response from the solution of eqs. (1) with an appropriate complex inductance. Notice that the agreement is excellent, without adjustable parameters, except $R(\omega)$. The eigenfrequencies do not depend on $R(\omega)$, but the intensities are.

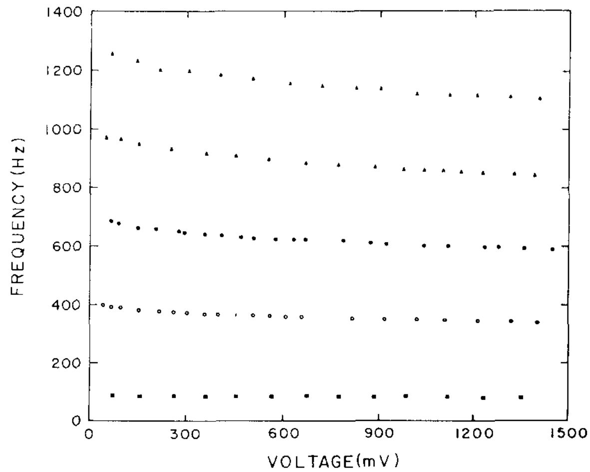

Fig. 4. Softening of the first five low-frequency states, as a function of the source rms voltage.

3. Results

The amplitude of the waves can be measured in all sites for any state. In fig. 3 we show the spatial distribution of amplitudes for the first six eigenstates of fig. 2. The state at 80 Hz is a truly acoustic mode with a wavelength four times the size of the lattice, and resembles the first normal mode of a membrane. This resemblance is to be expected, since for these long wavelengths the waves do not "see" the discreteness of the lattice. Analogously, the following eigen-states correspond to the excited states of a membrane, except that the higher the frequency, the more important the quasicrystalline array of bonds is. In particular, when the wavelength is comparable to the intersite distance, the amplitude starts to be localized in certain small regions, as a result of the quasicrystallinity. The peaks seen at the corners of figs. 3(e) and 3(f) are finite size effects, but the peaking around a five-fold coordinated atom in the center is not, since it is isolated by a line of nodes and of internal structure corresponding to the computed calculations for tiles of different sizes.

In fig. 4 we show the variation of the first five eigenvalues when the source voltage is increased. Notice that there is a softening of the modes except for the lowest frequency one. This softening is due to anharmonic effects, caused by the magnitude of the fields. In order to examine such a behavior theoretically, one has to consider Kirchhoff's laws in differential form:

$$C\frac{\mathrm{d}^2V_n}{\mathrm{d}t^2} + \sum_m \frac{1}{L}(V_n - V_m) = 0; \tag{3}$$

where the inductance $L = L_0 + l\Delta V + \mathrm{i}(r/\omega)\Delta V$, and $L_0 = 7.7$ mH, $l = 4.5$ mH/V and $r = 5.6$ Ω/V. Expanding eq. (3) as a Taylor series, we obtain

$$C\frac{\mathrm{d}^2V_n}{\mathrm{d}t^2} + \frac{1}{L_0}\sum_m \left\{(V_n - V_m) - \left(\frac{l + \mathrm{i}\frac{r}{\omega}}{L_0}\right)\times(V_n - V_m)^2 + \left(\frac{l + \mathrm{i}\frac{r}{\omega}}{L_0}\right)^2 \times(V_n - V_m)^3 - \dots\right\} = 0. \tag{4}$$

This equation represents a system of 137 non-linear coupled differential equations and is similar to the equation of motion for anharmonic phonons, which has no analytic solution. Therefore, our circuit provides a direct way of knowing the solutions of these non-linear systems of equations.

4. Discussion

In order to estimate the magnitude of the anharmonic effects one could write eq. (4) neglecting the coupling between different oscillators

$$\frac{d^2V}{dt^2} + \omega_0^2 V - \omega_0^2 \epsilon V^2 + \omega_0^2 \epsilon^2 V^3 - \ldots = 0, \qquad (5)$$

where $\epsilon = (l + i(r/\omega))/L_0$. Equation (5) could be solved by successive approximations and the frequency shift to first order is [13]

$$\omega - \omega_0 \simeq -\frac{A^2\epsilon^2\omega_0}{24}, \qquad (6)$$

where A is the amplitude of the unperturbed oscillator. Alternatively, one could explain the softening of the modes by looking at the expression for the unperturbed frequencies $\omega_0 = 1/\sqrt{LC}$, since, when the voltage increases, the inductances also increase, and the frequencies are lowered. The explicit form in which this softening takes places depends upon the topology of the lattice.

5. Conclusion

As a conclusion we could state that we have presented a useful method to study wave propagation in two-dimensional lattices, and we have built a device to look at wave interference in a Penrose lattice. This device is versatile and could be used for numerous studies, as selective conductance, anisotropy of the current flux and resonance phenomena, just to mention a few.

Despite of the small size of the network, the effects of quasiperiodicity and non-linearity are detectable on the eigenmodes of the system. We have demonstrated that the non-linear effects, when one increases the voltage, can be explained by anharmonic terms in the equations, and that in general these terms produce a softening of the modes of the harmonic system.

This device would be ideal to simulate interesting and complicated situations, as the dispersion of light by a rough metallic surface, since in that case local capacitances and inductances are formed, as in this circuit.

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