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Low-dimensional BEC

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The Bose-Einstein condensation (BEC) temperature T_c of Cooper pairs (CPs) created from a general interfermion interaction is determined for a linear, as well as the usually assumed quadratic, energy vs center-of-mass momentum dispersion relation. This explicit T_c is then compared with a widely applied implicit one of Wen & Kan (1988) in $d = 2 + \epsilon$ dimensions, for small ϵ, for a geometry of an infinite stack of parallel (e.g., copper-oxygen) planes as in, say, a cuprate superconductor, and with a new result for linear-dispersion CPs. The implicit formula gives T_c values only slightly lower than those of the explicit formula for typical cuprate parameters.

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Bose-Einstein condensation (BEC) of Cooper pairs (CPs) can lead to a phase transition (even in 2D) in any many-fermion system dynamically capable of forming CPs. This transition could be the origin of "exotic" superconductivity in the quasi-2D cuprates and in the quasi-1D organo-metallic (Bechgaard) salts, as well as of the superfluidity in liquid ^{3}He or in trapped Fermi gases in 3D.

The familiar BEC formula for the transition temperature is

$$T_c \simeq 3.31\hbar^2 n_B^{2/3}/m_B k_B, \tag{1}$$

with n_B the number density of bosons of mass m_B and k_B the Boltzmann constant. This is a special case of the more general expression[1] valid for any space dimensionality $d > 0$ and any boson dispersion relation $\varepsilon_K = C_s K^s$

with $s > 0$ and C_s constant, given by the *explicit* T_c-formula

$$T_c = \frac{C_s}{k_B}\left[\frac{s\,\Gamma(d/2)\,(2\pi)^d n_B}{2\pi^{d/2}\,\Gamma(d/s) g_{d/s}(1)}\right]^{s/d} . \tag{2}$$

If $\mu(T)$ is the boson chemical potential and $e^{\mu(T)/k_BT} \equiv z$ the fugacity, $g_\sigma(z) \equiv \sum_{l=1}^{\infty} z^l/l^\sigma$ are the Bose integrals. For $z = 1$ and $\sigma \geq 1$ the $g_\sigma(1)$ is just $\zeta(\sigma)$, the Riemann Zeta-function of order σ which is finite for $\sigma > 1$ and infinite for $\sigma = 1$, while the series $g_\sigma(1)$ diverges for all $\sigma \leq 1$. For $s = 2$, $C_2 = \hbar^2/2m_B$, and since $\zeta(3/2) \simeq 2.612$, this leads to the usual BEC T_c-formula (1). Since $g_{d/2}(1)$ diverges for all $d/2 \leq 1$, $T_c = 0$ for all $d \leq 2$. This follows from the boson number equation

$$N = N_0(T) + \sum_{\mathbf{K}\neq 0}\left[e^{\{\varepsilon_K - \mu(T)\}/k_BT} - 1\right]^{-1} \tag{3}$$

where $N_0(T)$ is the number of bosons in the $K = 0$ state. At $T = T_c$ both $N_0(T_c)$ *and* the boson chemical potential $\mu(T_c)$ virtually vanish so that replacing

$$\sum_{\mathbf{k}\neq 0} \longrightarrow (L/2\pi)^d \int_{0+} d^dk = (L/2\pi)^d \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{0+}^{\infty} dk\, k^{d-1} \tag{4}$$

in (3) eventually yields (2) where $n_B \equiv N/L^d$.

The fact that a CP can have a *linear* ($s = 1$), as opposed to the usual quadratic ($s = 2$), dispersion relation was mentioned as far back as 1964 by Schrieffer,[2] p. 33, for the BCS model interaction in 3D. This was recently confirmed[3] to be the case for both 2D and 3D under a very general inter-fermion interaction for any coupling provided the fermion number-density is nonzero, i.e., in the presence of a Fermi sea. The CP dispersion relation becomes quadratic *only in the extremely dilute* (or vacuum) *limit* where the CPs are just the so-called "local pairs." For any sizeable fermion density the nonnegative CP *excitation energy* $\varepsilon_K \equiv \Delta_0 - \Delta_K$ behaves like $\simeq a(d)\hbar v_F K$, where Δ_K (not to be confused with the BCS gap Δ) is the (positive) binding energy of a CP of center-of-mass momentum (CMM) $\hbar K$, $v_F \equiv \hbar k_F/m$ and k_F the Fermi velocity and wavenumber, respectively, m the fermion effective mass, while $a(d) \equiv 2/\pi$ and $1/2$ in 2D and 3D, respectively, precisely as established[4] previously for the BCS model interaction. For linear dispersion $s = 1$, $C_1 = a(d)\hbar v_F$, $T_c = 0$ from (2) for all $d \leq 1$ only—and $T_c > 0$ for all $d > 1$, which is precisely the range of dimensionalities for all known superconductors if one includes the quasi-1D organo-metallic Bechgaard salts.[5] Using the interpolation $a(d) = (7/2 - 6/\pi) + (8/\pi - 13/4)d + (3/4 - 2/\pi)d^2$, which correctly reduces to 1, $2/\pi$ and $1/2$ in 1D, 2D and 3D, respectively, Fig. 1 graphs (2) for $s = 1$ and 2 (in units of the Fermi temperature T_F) *vs* d—if one imagines all the fermions in the initial, interactionless many-fermion system paired into CPs of mass $m_B = 2m$. The particle number density of the original fermions is $n \equiv k_F^d/2^{d-2}\pi^{d/2} d\,\Gamma(d/2)$ and equals $2n_B$, and we have used $\hbar^2 k_F^2/2m = \frac{1}{2}mv_F^2 = E_F \equiv k_B T_F$. These curves are *upper bounds* to the T_c from a more realistic model[6] where chemical equilibrium allows only a *fraction* of all fermions to be actually bound into pairs.

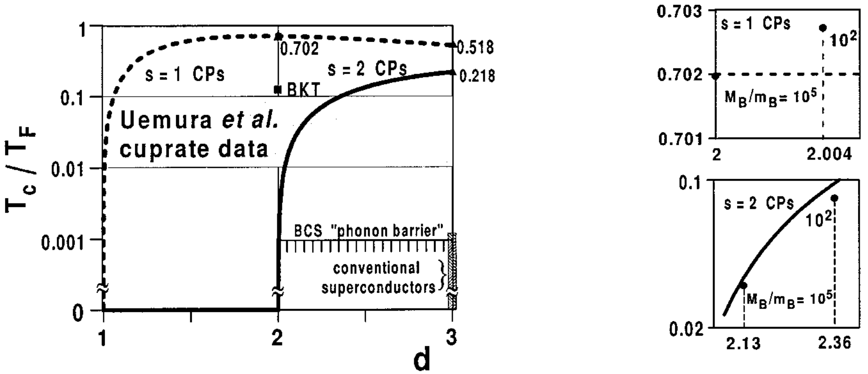

Fig. 1. **Left:** dimensionality, d, dependence of the critical BEC transition temperature T_c according to *explicit* (2) (in units of the Fermi temperature T_F) as explained in text. Lower (full) curve is for $s = 2$; upper (dashed) curve for $s = 1$. Shaded areas refer to empirical data from Ref. [7]. **Right:** dots refer to results from *implicit* (6) below, as explained just after (7).

A rather general interfermion interaction is the S-wave attractive separable potential whose double Fourier transform is

$$V_{pq} = -(v_0/L^2)g_p g_q. \tag{5}$$

Here L is the size of the "box" confining the many-fermion system, $v_0 \geq 0$ is the interaction strength and g_p given, e.g.,[8] by $(1 + p^2/p_0^2)^{-1/2}$ where p_0 is the inverse range of the potential. Hence, $p_0 \to \infty$ implies $g_p = 1$ and corresponds to the attractive contact (or delta) potential $V(r) = -v_0\delta(\mathbf{r})$, while $p_0 = k_F$ implies a range of order of the average interfermion spacing, etc. If $g_p = \theta(\hbar\omega_D + \mu_F - p^2/2m)$, with $\theta(x)$ the unit step function, (5) becomes the BCS model interaction where ω_D is the Debye frequency and μ_F the fermionic chemical potential that becomes E_F for $T = 0 = v_0$.

Using a *renormalized CP equation*[3] whose coupling depends *only* on the two-body binding energy B_2,[9] the CP excitation energy $\varepsilon_K \equiv \Delta_0 - \Delta_K$ for zero range was obtained numerically[3] as an exact curve that for very small B_2/E_F is virtually linear, i.e., $\varepsilon_K \rightarrow 2\hbar v_F K/\pi$. It is *only* in the dilute limit (v_F or $E_F \rightarrow 0$) that ε_K tends asymptotically to the exact quadratic $\hbar^2K^2/2(2m)$ for any coupling. Assuming $n_B = n/2$ and $m_B = 2m$ to introduce the temperature scale T_F as before, Fig. 2 shows the BEC T_c's of a pure gas of *unbreakable* CPs. Significantly, T_c is *no longer* zero in 2D—as would be predicted in a BEC picture by a quadratic relation appropriate for "local-pair" CPs in vacuum, a result that wrongly suggests that BEC cannot apply for quasi-2D cuprate superconductors.

More accurate BEC T_c's should include refinements such as non-S-wave interactions, allowing for *unpaired fermions* in a more realistic binary boson-fermion mixture model[6]—and most importantly, CP-fermion interactions that link[10,11] the BEC condensate fraction temperature-dependence with that of the BCS fermionic energy gap, among other corrections.

The linear dispersion relation of a CP *should not* be confused with the linear dispersion of Anderson-Bogoliubov-Higgs (ABH) many-body excitation phonon-like modes. Collective modes in a superconductor were studied since the late 1950's by several workers. A more recent treatment for 1D, 2D and 3D is available[12] which confirms the linear ABH form $\hbar v_F K/\sqrt{d}$ for d =1, 2 or 3 in the zero-coupling limit. Our CPs are taken as "bosonic" even though they do *not* obey (Ref. [2] p. 38) Bose commutation relations. This is because for a given K they have *indefinite* occupation number since for fixed K there are (after the thermodynamic limit) an indefinitely large number of allowed (relative wavenumber) k values, so that—for any coupling and thus any degree of overlap between them—CPs do in fact obey the Bose-Einstein distribution from which BEC is determined. By contrast, ABH phonons (like photons or plasmons, etc.) *cannot* suffer a BEC as their number is always indefinite. The number of CPs, on the other hand, is *fixed* at half the number of (pairable) fermions if all of these are imagined paired at a given temperature and coupling.

To model cuprate superconductors consider the bosons confined to an infinite set of planes stacked along the z-direction, parallel to each other with equal spacing c between adjacent planes. The BEC transition temperature formula, for $s = 2$ bosons in each plane, is the *implicit* T_c-formula[13]

$$k_BT_c = \frac{2\pi n_{B-3D}\hbar^2 c}{m_B \ln[M_B c^2 k_B T_c \nu(t_c)/\hbar^2]}, \tag{6}$$

and is valid in $2 + \epsilon$ dimensions where ϵ is small and given by

$$\epsilon \simeq 2[\ln(n_{B-3D} M_B c^3/m_B)]^{-1}. \tag{7}$$

This expressly vanishes as $M_B c^3 \rightarrow \infty$, as it should. Here $n_{B-3D} \equiv N/L^3$ while $\nu(t) = 1 - t + O(t^2) \simeq 1$ if $t << 1$, where $t \equiv \hbar^2/M_B c^2 k_B T =$

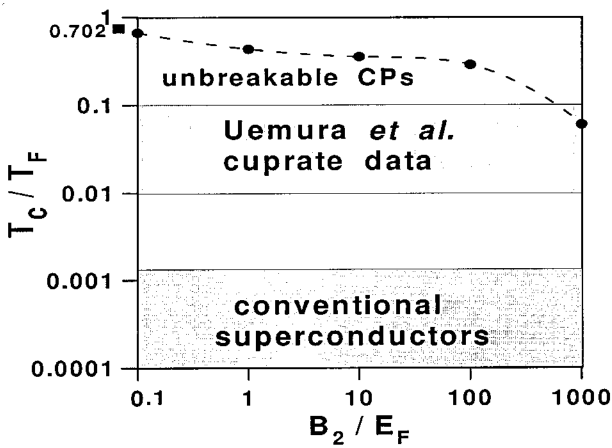

Fig. 2. 2D critical BEC temperatures T_c (in units of T_F) for five coupling values B_2/E_F for a pure boson gas of unbreakable CPs, determined by the $N_0(T_c) = 0 = \mu(T_c)$ solution of (3) inserting exact numerical CP dispersion curves, using (4). The value 0.702 marked with a square corresponds to the T_c/T_F value at zero coupling (see also Ref. [1]) and contrasts with the well-known result $T_c/T_F \equiv 0$ in 2D for infinite coupling where $\varepsilon_K = \hbar^2K^2/2(2m)$ exactly. Shaded areas as in Fig. 1.

$\hbar^2/2mc^2(M_B/m_B)k_BT$. Using $\hbar^2/mk_B = 88,419$ K Å^2 with m the electron mass, and c = 12 Å, this inequality is well satisfied for the higher cuprate transition temperatures, today ranging up to 164 K, since $T_c >> 307\text{K}/(M_B/m_B)$ as typically M_B/m_B can range from 10^2 to 10^5. Clearly, for $M_Bc \to 0$ (infinitely separated planes and/or perfect confinement to the z-direction in each plane) T_c vanishes as it should in 2D. As state, this T_c-equation is *implicit* or transcendental, unlike the simpler *explicit* T_c equations (1) and (2), and has been used for varied purposes by numerous authors[10,14,15]—though only for *quadratic* dispersion bosons.

We have generalized (6) for any $s > 0$ and found

$$k_BT_c = C_s \left[2\pi n_{B-3D} sc/\Gamma(2/s) g_{2/s}(e^{-\hbar^2/M_Bc^2k_BT_c})\right]^{s/2}. \quad (8)$$

Thus, an exact (again, implicit) equation for T_c is obtained for any $s > 0$. For $s = 2$ and $C_2 = \hbar^2/2m_B$ we recover (6). In Fig. 1 (right) we plot results for both values of s as points, for $M_B/m_B = 10^2$ and 10^5. They can be seen to differ very slightly from the results for $s = 2$ or $s = 1$ bosons in $d = 2 + \epsilon$ dimensions, with ϵ small, that came directly from (2) which, moreover, is valid for all $d > 0$.

To compare our results, consider the Berezinskii-Kosterlitz-Thouless[16] transition temperature formula

$$k_B T_c^{BKT} = \frac{\pi}{2} \frac{\hbar^2 n_B}{m_B} \tag{9}$$

valid in 2D, and assume as before that $n_B = n/2 \equiv k_F^2/4\pi$ and $m_B = 2m$. This gives $T_c^{BKT}/T_F = 1/8 = 0.125$ and is displayed as a square in Fig. 1.

In conclusion, BEC T_c's related to a pure gas of unbreakable composite linear-dispersion bosons were calculated in $d = 2$ for Cooper pairs formed via a general separable potential and whose coupling for any CMM is characterized solely by its two-body binding energy in vacuum. A T_c-formula valid for *any* dispersion relation of the form $\varepsilon_K = C_s K^s$ with $s > 0$ and C_s constant is deduced for the BEC of a pure gas of unbreakable bosons confined to move in infinitely-many identical planes parallel to each other. It is a peculiar *implicit* T_c-equation valid in $2 + \epsilon$ where ϵ is small and vanishes, as it must, when the inter-plane spacing or the boson mass in the direction perpendicular to the planes diverges. However, for either $s = 2$ or 1 we find results that are just slightly different than those of the *explicit* BEC T_c-formula which is valid more generally for any $d > 0$.

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