

Introduction to Renormalization in Field Theory

Ling-Fong Li

Carnegie Mellon University, Pittsburgh, PA. USA

and

Chongqing University of Posts & Telecommunications,
Chongqing, China

Contents

1	Introduction	1
2	Renormalization Schemes	3
2.1	Conventional renormalization	3
2.2	BPH renormalization	11
3	Power counting and Renormalizability	18
3.1	Theories with fermions and scalar fields	18
3.2	Renormalization of Composite Operators	24
3.3	Symmetry and Renormalization	27

Abstract

A simple introduction of renormalization in quantum field theory is discussed. Explanation of concepts is emphasized instead of the technical details.

1 Introduction

Many people who have studied quantum field theory find the most difficult part is the theory of renormalization. The relativistic field theory is full of infinities which need to be taken care of before the theoretical predictions can be compared with experimental measurements. These infinities look

formidable at first sight. It is remarkable that over the years a way has been found to make sense of these apparently divergent theories ([1],[2]).

The theory of renormalization is a prescription which consistently isolates and removes all these infinities from the physically measurable quantities. Note that the need for renormalization is quite general and is not unique to the relativistic field theory. For example, consider an electron moving inside a solid. If the interaction between electron and the lattice of the solid is weak enough, we can use an effective mass m^* to describe its response to an externally applied force and this effective mass is certainly different from the mass m measured outside the solid. Thus the electron mass is changed (renormalized) from m to m^* by the interaction of the electron with the lattice in the solid. In this simple case, both m and m^* are measurable and hence finite. For the relativistic field theory, the situation is the same except for two important differences. First the renormalization due to the interaction is generally infinite (corresponding to the divergent loop diagrams). These infinities, coming from the contribution of high momentum modes are present even for the cases where the interactions are weak. Second, there is no way to switch off the interaction between particles and the quantities in the absence of interaction, bare quantities, are not measurable. Roughly speaking, the program of removing the infinities from physically measurable quantities in relativistic field theory, the renormalization program, involves shuffling all the divergences into bare quantities. In other words, we can redefine the unmeasurable quantities to absorb the divergences so that the physically measurable quantities are finite. The renormalized mass which is now finite can only be determined from experimental measurement and can not be predicted from the theory alone.

Eventhough the concept of renormalization is quite simple, the actual procedure for carrying out the operation is quite complicated and intimidating. In this article, we will give a bare bone of this program and refer interested readers to more advanced literature ([3],[4]). Note that we need to use some regularization procedure ([5]) to make these divergent quantities finite before we can do mathematically meaningful manipulations. We will not discuss this part in the short presentation here. Also note that not every relativistic field theory will have this property that all divergences can be absorbed into redefinition of few physical parameters. Those which have this property are called *renormalizable* theories and those which don't are called unrenormalizable theories. This has become an important criteria for choosing a right theory because we do not really know how to handle the *unrenormalizable* theory.

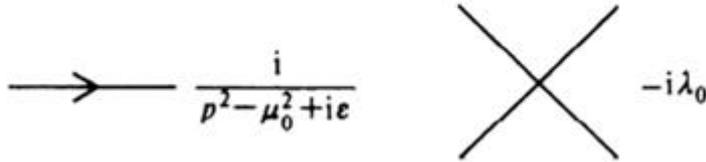


Fig 1 Feynman rule for $\lambda\phi^4$ theory

2 Renormalization Schemes

There are two different methods to carry out the renormalization program, i) conventional renormalization which is more intuitive but mathematically complicated, ii) BPH renormalization which is simple to describe but not so transparent ([3]). These two methods are in fact complementary to each other and it is very useful to know both.

2.1 Conventional renormalization

We will illustrate this scheme in the simple $\lambda\phi^4$ theory where the Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

with

$$\mathcal{L}_0 = \frac{1}{2} [(\partial_\mu \varphi_0)^2 - \mu_0^2 \varphi_0^2]$$

and

$$\mathcal{L}_1 = -\frac{\lambda_0}{4!} \varphi_0^4$$

Here $\mu_0, \lambda_0, \varphi_0$ are bare mass, bare coupling constant and bare field respectively. The propagator and vertex of this theory are given below, Here p is the momentum carried by the line and μ_0^2 is the bare mass term in \mathcal{L}_0 .

The two point function (propagator) defined by

$$i\Delta(p) = \int d^4x e^{-ip \cdot x} \langle 0 | T(\varphi_0(x) \varphi_0(0)) | 0 \rangle$$

can be written in terms of one-particle-irreducible, or 1PI (those graphs which can not be made disconnected by cutting any one internal line) as a

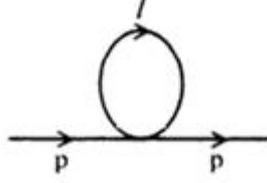


Fig 2. 1-loop 2-point function

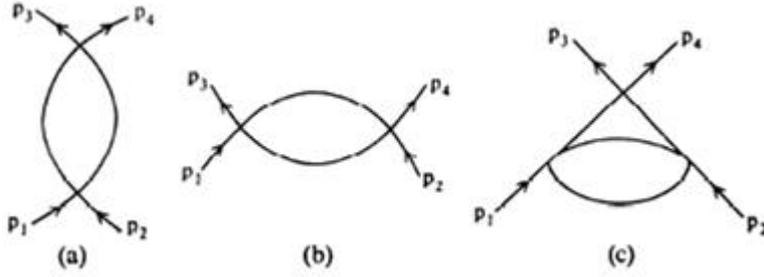


Fig.3 One-loop 4-point functions

geometric series

$$i\Delta(p) = \frac{i}{p^2 - \mu_0^2 + i\varepsilon} + \frac{i}{p^2 - \mu_0^2 + i\varepsilon} (-i\Sigma(p^2)) \frac{i}{p^2 - \mu_0^2 + i\varepsilon} + \dots \quad (1)$$

$$= \frac{i}{p^2 - \mu_0^2 - \Sigma(p^2) + i\varepsilon}$$

Here $\Sigma(p^2)$ is the IPI self energy graph. In one-loop, the 1PI divergent graphs are

For the self energy the contribution is,

$$-i\Sigma(p^2) = -\frac{i\lambda_0}{2} \int \frac{d^4l}{(2\pi)^4} \frac{i}{l^2 - \mu_0^2 + i\varepsilon} \quad (2)$$

which diverges quadratically and for the 4-point functions we have

$$\Gamma_a = \Gamma(p^2) = \Gamma(s) = \frac{(-i\lambda_0)^2}{2} \int \frac{d^4l}{(2\pi)^4} \frac{i}{l^2 - \mu_0^2 + i\varepsilon} \frac{i}{(l-p)^2 - \mu_0^2 + i\varepsilon} \quad (3)$$

$$\Gamma_b = \Gamma(t), \quad \Gamma_c = \Gamma(u)$$

Here

$$s = p^2 = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2,$$

are the Mandelstam variables and $\Gamma(s)$ diverges logarithmically.

One important feature to note about these integrals is that when we differentiate them with respect to external momenta, the integral will become more convergent. For example, if we differentiate $\Gamma(p^2)$ with respect to p^2 , one finds

$$\begin{aligned}\frac{\partial}{\partial p^2}\Gamma(p^2) &= \frac{1}{2p^2}p_\mu \frac{\partial}{\partial p_\mu}\Gamma(p^2) \\ &= \frac{\lambda_0^2}{p^2} \int \frac{d^4l}{(2\pi)^4} \frac{(l-p) \cdot p}{l^2 - \mu_0^2 + i\varepsilon} \frac{1}{[(l-p)^2 - \mu_0^2 + i\varepsilon]^2}\end{aligned}$$

which is finite. This means that the divergences will reside only in the first few terms in a Taylor expansion in the external momenta of the Feynman diagram. In our case, we can write

$$\Gamma(s) = \Gamma(0) + \bar{\Gamma}(s)$$

where $\Gamma(0)$ is logarithmic divergent and $\bar{\Gamma}(s)$, which is the sum of all higher derivative terms, is finite. In other words, the finite part $\bar{\Gamma}(s)$ corresponds to subtracting the divergent part $\Gamma(0)$ from $\Gamma(s)$ and is sometimes referred to as the *subtraction*.

Mass and wavefunction renormalization

The self energy contribution in Eq (2) is quadratically divergent. To isolate the divergences we use the Taylor expansion around some arbitrary value μ^2 ,

$$\Sigma(p^2) = \Sigma(\mu^2) + (p^2 - \mu^2)\Sigma'(\mu^2) + \tilde{\Sigma}(p^2)$$

where $\Sigma(\mu^2)$ is quadratically divergent, and $\Sigma'(\mu^2)$ is logarithmically divergent and $\tilde{\Sigma}(p^2)$ is finite. The finite part $\tilde{\Sigma}(p^2)$ will have the property,

$$\tilde{\Sigma}(\mu^2) = \tilde{\Sigma}'(\mu^2) = 0 \quad (4)$$

Note that self-energy in 1-loop has the peculiar feature that it is independent of the external momentum p^2 and the Taylor expansion has only one term, $\Sigma(\mu^2)$. However, the higher loop contribution does depend on the external momentum and the Taylor expansion is non-trivial. The propagator in Eq (1) is then,

$$i\Delta(p) = \frac{i}{p^2 - \mu_0^2 - \Sigma(\mu^2) - (p^2 - \mu^2)\Sigma'(\mu^2) - \tilde{\Sigma}(p^2) + i\varepsilon}$$

The physical mass is defined as the position of the pole in the propagator. Since up to this point μ^2 is arbitrary, we can choose it to satisfy the relation,

$$\mu_0^2 + \Sigma(\mu^2) = \mu^2, \quad (5)$$

Then

$$i\Delta(p) = \frac{i}{(p^2 - \mu^2) [1 - \Sigma'(\mu^2)] - \tilde{\Sigma}(p^2) + i\epsilon}$$

and using Eq (4) we see that $\Delta(p)$ has a pole at $p^2 = \mu^2$. Thus μ^2 is the **physical mass** and is related to the bare mass μ_0^2 in Eq (5). This is the *mass renormalization*. Since $\Sigma(\mu^2)$ is divergent, the bare mass μ_0^2 must also be divergent so that the combination $\mu_0^2 + \Sigma(\mu^2)$ is finite and measurable. In other words, the bare mass μ_0^2 has to diverge in such a way that its divergence cancels the divergent loop correction to yield a finite result. It amounts to shuffling the infinities to unobservable quantities like bare mass μ_0^2 . This is the part in renormalization theory which is very difficult to comprehend at the first sight. Nevertheless it is logically consistent and the rules are very precise. Furthermore, the results after the renormalization have been successfully checked by experiments. This gives us confidence about the validity of renormalization theory.

To remove the divergent quantity $\Sigma'(\mu^2)$ we note that in 1-loop both $\Sigma'(\mu^2)$, $\tilde{\Sigma}(p^2)$ are of order λ_0 , for convenience, we can make the approximation,

$$\tilde{\Sigma}(p^2) \simeq [1 - \Sigma'(\mu^2)] \tilde{\Sigma}(p^2) + O(\lambda_0^2)$$

and write the propagator as

$$i\Delta(p) = \frac{iZ_\varphi}{(p^2 - \mu^2) - \tilde{\Sigma}(p^2) + i\epsilon}$$

where

$$Z_\varphi = [1 - \Sigma'(\mu^2)]^{-1} \simeq 1 + \Sigma'(\mu^2) + O(\lambda_0^2) \quad (6)$$

Now the divergence is shuffled into the multiplicative factor Z_φ which can be removed by defining a renormalized field φ as,

$$\varphi = Z_\varphi^{-1/2} \varphi_0 \quad (7)$$

The propagator for the renormalized field is then

$$\begin{aligned} i\Delta_R(p) &= \int d^4x e^{-ip \cdot x} \langle 0 | T(\varphi(x) \varphi(0)) | 0 \rangle \\ &= iZ_\varphi^{-1} \Delta(p) = \frac{i}{(p^2 - \mu^2) - \tilde{\Sigma}(p^2) + i\epsilon} \end{aligned} \quad (8)$$

and it is completely finite. Z_φ is usually called the *wavefunction renormalization constant*. Thus another divergence is shuffled into the bare field operator φ_0 which is also not measurable.

The new renormalized field operator φ should also be applied to the renormalized higher point Green's functions,

$$\begin{aligned} G_R^{(n)}(x_1, x_2, \dots, x_n) &= \langle 0 | T(\varphi(x_1) \varphi(x_2) \dots \varphi(x_n)) | 0 \rangle \\ &= Z_\varphi^{-n/2} \langle 0 | T(\varphi_0(x_1) \varphi_0(x_2) \dots \varphi_0(x_n)) | 0 \rangle \\ &= Z_\varphi^{-n/2} G_0^{(n)}(x_1, x_2, \dots, x_n) \end{aligned}$$

Here $G_0^{(n)}(x_1, x_2, \dots, x_n)$ is the unrenormalized n -point Green's function. Or in momentum space

$$G_R^{(n)}(p_1, p_2, \dots, p_n) = Z_\varphi^{-n/2} G_0^{(n)}(p_1, p_2, \dots, p_n)$$

where

$$(2\pi)^4 \delta^4(p_1 + \dots + p_n) G_R^{(n)}(p_1, \dots, p_n) = \int \left(\prod_{i=1}^n dx_i^4 e^{-ip_i \cdot x_i} \right) G_R^{(n)}(x_1, \dots, x_n)$$

Similarly for $G_0^{(n)}(p_1, p_2, \dots, p_n)$. To go from the connected Green's functions to the 1PI (amputated) Green's functions, we need to eliminate the one-particle reducible diagrams, and also to remove the propagators $i\Delta_R(p_i)$ for the external lines in 1PI Green's function $G_R^{(n)}(p_1, \dots, p_n)$. As a result the relation between 1PI Green's functions are of the form,

$$\Gamma_R^{(n)}(p_1, p_2, \dots, p_n) = Z_\varphi^{n/2} \Gamma_0^{(n)}(p_1, p_2, \dots, p_n)$$

Note that the relations in Eq (4) are direct consequence of the Taylor expansion around the point $p^2 = \mu^2$ which is totally arbitrary. From the form of the renormalized propagator in Eq (8), we see that Eq (4) are equivalent to the relations

$$\Delta_R^{-1}(\mu^2) = 0, \quad \left. \frac{d}{dp^2} \Delta_R^{-1}(p^2) \right|_{p^2=\mu^2} = 1$$

If we have chosen some other point, e.g. $p^2 = 0$ for the Taylor expansion, the finite part $\tilde{\Sigma}_1(p^2)$ will have the properties

$$\tilde{\Sigma}_1(0) = \tilde{\Sigma}'_1(0) = 0 \tag{9}$$

Or in terms of renormalized propagator,

$$\Delta_R^{-1}(0) = -\mu^2, \quad \left. \frac{d}{dp^2} \Delta_R^{-1}(p^2) \right|_{p^2=0} = 1$$

Sometimes in the renormalization prescription we replace the statement " Taylor expansion around $p^2 = \mu^2$, or $p^2 = 0$ " by relations expressed, in Eq (4,9), called the *renormalization conditions*. One important feature to keep in mind is that in carrying out the renormalization program there is an arbitrariness in choosing the points for the Taylor expansion. Different renormalization schemes seem to give rise to different looking relations. However, if these renormalization schemes make any sense at all, the physical laws which are relations among physically measurable quantities should be the same regardless of which scheme is used. This is the basic idea behind the *renormalization group equations* ([6]).

Coupling constant renormalization

The basic coupling in $\lambda\varphi^4$ theory is the 4-point function which in 1-loop has the form, before renormalization,

$$\Gamma_0^{(4)}(s, t, u) = -i\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u)$$

where last three terms are logarithmic divergent. We will remove these divergences by the redefinition of the coupling constant. Note that the physical coupling constant is measured in terms of two-particle scattering amplitude which is essentially 1PI 4-point Green's function $\Gamma_R^{(4)}(s, t, u)$ which is a function of the kinematical variables, s, t and u . For convenience, we can choose the symmetric point,

$$s_0 = t_0 = u_0 = \frac{4\mu^2}{3}$$

to define the coupling constant,

$$\Gamma_R^{(4)}(s_0, t_0, u_0) = -i\lambda$$

where λ is the renormalized coupling constant. Since $\Gamma(s)$ is only logarithmically divergent, we can isolate the divergence in one term in the Taylor expansion,

$$\Gamma(s) = \Gamma(s_0) + \tilde{\Gamma}(s)$$

where $\tilde{\Gamma}(s)$ is finite and

$$\tilde{\Gamma}(s_0) = 0$$

Then

$$\Gamma_0^{(4)}(s, t, u) = -i\lambda_0 + 3\Gamma(s_0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)$$

We can isolate the divergence by combining the first two term and define the vertex renormalization constant Z_λ by

$$-iZ_\lambda\lambda_0 = -i\lambda_0 + 3\Gamma(s_0)$$

Then

$$\Gamma_0^{(4)}(s, t, u) = -iZ_\lambda\lambda_0 + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)$$

The renormalized 4-point 1PI is then

$$\begin{aligned}\Gamma_R^{(4)}(s, t, u) &= Z_\varphi^2\Gamma_0^{(4)}(s, t, u) \\ &= -iZ_\lambda Z_\varphi^2\lambda_0 + Z_\varphi^2\left[\tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)\right]\end{aligned}\quad (10)$$

We now define the renormalized coupling constant λ as

$$\lambda = Z_\lambda Z_\varphi^2\lambda_0 \quad (11)$$

and from Eq (6) we see that

$$Z_\varphi = 1 + O(\lambda_0)$$

Also $\tilde{\Gamma}$ is of order of λ_0^2 . The renormalized 4-point 1PI can be put into the form,

$$\Gamma_R^{(4)}(s, t, u) = \lambda + \left[\tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)\right] + O(\lambda_0^3)$$

Assuming that the coupling constant λ is measured in the scattering experiment and is finite, we see that this 4-point function is completely free of divergences. Eq (11) shows that the renormalization of coupling constant involves wavefunction renormalization in addition to the vertex correction.

For the renormalization of connected Green's functions, we need to add one-particle reducible diagrams and attach propagators for the external lines. We want to show that the renormalized Green functions when expressed in terms of renormalized quantities are completely finite. We start with the unrenormalized Green's function of the form,

$$G_0^{(4)}(p_1, \dots, p_4) = \prod_{j=1}^4 \Delta^{(0)}(p_j) \{-i\lambda_0 + 3\Gamma(s_0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)\} \quad (12)$$

$$+ (-i\lambda_0) \sum_{k=1}^4 [-i\Sigma(p_k^2) i\Delta^{(0)}(p_k)] \quad (13)$$

where

$$\Delta^{(0)}(p_j) = \frac{1}{p_j^2 - \mu_0^2 + i\varepsilon}$$

is the zeroth order bare propagator and the last terms here are coming from the diagrams of the following type,

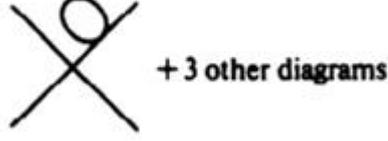


Fig 4, 1-particle reducible 4 point function

We can combine the first term and the last terms in $G_0^{(4)}(p_1, \dots, p_4)$ to get

$$\begin{aligned} (-i\lambda_0) \left\{ 1 + \sum_{k=1}^4 [\Sigma(p_k^2) \Delta^{(0)}(p_k)] \right\} &\simeq (-i\lambda_0) \left[\prod_{k=1}^4 \frac{1}{1 - \Sigma(p_k^2) \Delta^{(0)}(p_k)} \right] + O(\lambda_0^3) \\ &= (-i\lambda_0) \prod_{k=1}^4 \left\{ [\Delta^{(0)}(p_k)]^{-1} \frac{1}{[p_k^2 - \mu_0^2 - \Sigma(p_k^2)]} \right\} = (-i\lambda_0) \prod_{k=1}^4 \left\{ [\Delta^{(0)}(p_k)]^{-1} \Delta(p_k) \right\} \end{aligned}$$

where

$$\Delta(p_k) = \frac{1}{[p_k^2 - \mu_0^2 - \Sigma(p_k^2)]}$$

Since the difference between $\Delta(p_k)$ and $\Delta^{(0)}(p_k)$ is higher order in λ_0 , we can make the approximation for the rest of the terms in Eq (12),

$$\prod_{j=1}^4 \Delta^{(0)}(p_j) \left[3\Gamma(s_0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \right] \simeq \prod_{j=1}^4 \Delta(p_j) \left[3\Gamma(s_0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \right]$$

The unrenormalized Green's function is then

$$\begin{aligned} G_0^{(4)}(p_1, \dots, p_4) &= \left[\prod_{j=1}^4 \Delta(p_j) \right] \left[-i\lambda_0 + 3\Gamma(s_0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \right] \\ &= \left[\prod_{j=1}^4 \Delta(p_j) \right] \Gamma_0^{(4)}(s, t, u) \end{aligned}$$

We now multiply the unrenormalized Green's function by the appropriate factor of Z_φ to get the renormalized one,

$$\begin{aligned} G_R^{(4)}(p_1, \dots, p_4) &= Z_\varphi^{-2} G_0^{(4)}(p_1, \dots, p_4) = Z_\varphi^{-2} \left[\prod_{j=1}^4 \Delta(p_j) \right] \Gamma_0^{(4)}(s, t, u) \\ &= Z_\varphi^{-2} \left[Z_\varphi^4 \prod_{j=1}^4 i\Delta_R(p_j) \right] Z_\varphi^{-2} \Gamma_R^{(4)}(s, t, u) \\ &= \left[\prod_{j=1}^4 i\Delta_R(p_j) \right] \Gamma_R^{(4)}(s, t, u) \end{aligned}$$

Thus we have removed all the divergences in the connected 4-point Green's function.

In summary, Green's functions can be made finite if we express the bare quantities in terms of the renormalized ones through the relations,

$$\varphi = Z_\varphi^{-1/2} \varphi_0, \quad \lambda = Z_\lambda^{-1} Z_\varphi^2 \lambda_0, \quad \mu^2 = \mu_0^2 + \delta\mu^2 \quad (14)$$

where $\delta\mu^2 = \Sigma(\mu^2)$. More specifically, for an n -point Green's function when we express the bare mass μ_0 and bare coupling λ_0 in terms of the renormalized mass μ and coupling λ , and multiply by $Z_\varphi^{-1/2}$ for each external line the result (the renormalized n -point Green's function) is completely finite,

$$G_R^{(n)}(p_1, \dots, p_n; \lambda, \mu) = Z_\varphi^{-n/2} G_0^{(n)}(p_1, \dots, p_n; \lambda_0, \mu_0, \Lambda)$$

where Λ is the cutoff needed to define the divergent integrals. This feature, in which all the divergences, after rewriting μ_0 and λ_0 in terms of μ and λ are aggregated into some multiplicative constants, is called being *multiplicatively renormalizable*.

Our discussion here contains some of the essential features in the renormalization program. To prove that the procedure we outline here will remove all the divergences in the theory is a very complicated mathematical undertaking and is beyond the scope of this simple introduction.

2.2 BPH renormalization

BPH renormalization (Bogoliubov and Parasiuk, Hepp) ([3]) is completely equivalent to the conventional renormalization but organized differently. We will illustrate this in the simple $\lambda\varphi^4$ theory.

Start from the unrenormalized Lagrangian,

$$\mathcal{L}_0 = \frac{1}{2} [(\partial_\mu \varphi_0)^2 - \mu_0^2 \varphi_0^2] - \frac{\lambda_0}{4!} \varphi_0^4$$

where all the quantities are unrenormalized. We can rewrite this in terms of renormalized quantities using Eq (14),

$$\mathcal{L}_0 = \mathcal{L} + \Delta\mathcal{L}$$

where

$$\mathcal{L} = \frac{1}{2} [(\partial_\mu \varphi)^2 - \mu^2 \varphi^2] - \frac{\lambda}{4!} \varphi^4 \quad (15)$$

has exactly the same form as the original Lagrangian, is called the *renormalized Lagrangian*, and

$$\Delta\mathcal{L} = \frac{(Z_\varphi - 1)}{2} [(\partial_\mu\varphi)^2 - \mu^2\varphi^2] + \frac{\delta\mu^2}{2} Z_\varphi\varphi^2 - \frac{\lambda(Z_\lambda - 1)}{4!}\varphi^4 \quad (16)$$

contains all the divergent constants, Z_φ , Z_λ , and $\delta\mu^2$, and is called the counterterm Lagrangian.

The BPH renormalization scheme consists of the following steps;

1. Start with renormalized Lagrangian given in Eq (15) to construct propagators and vertices.
2. Isolate the divergent parts of 1PI diagrams by Taylor expansion. Construct a set of counterterms $\Delta\mathcal{L}^{(1)}$ which is designed to cancel these one-loop divergences.
3. A new Lagrangian $\mathcal{L}^{(1)} = \mathcal{L} + \Delta\mathcal{L}^{(1)}$ is used to generate the 2-loop diagrams and to construct the counterterms $\Delta\mathcal{L}^{(2)}$ which cancels the divergences up to this order and so on, as this sequence of operations is iteratively applied.

The resulting Lagrangian is of the form,

$$\mathcal{L}^{(\infty)} = \mathcal{L} + \Delta$$

where the counterterm Lagrangian $\Delta\mathcal{L}$ is given by,

$$\Delta\mathcal{L} = \Delta\mathcal{L}^{(1)} + \Delta\mathcal{L}^{(2)} + \dots \Delta\mathcal{L}^{(n)} + \dots$$

We will now show that the counter term Lagrangian has the same structure as that in Eq (16).

Power Counting Method

This method will help to classify divergences systematically. For a given Feynman diagram, we define **superficial degree of divergence** D as the number of loop momenta in the numerator minus the number of loop momenta in the denominator. For illustration we will compute D in $\lambda\phi^4$ theory. Define

$$\begin{aligned} B &= \text{number of external lines} \\ IB &= \text{number of internal lines} \\ n &= \text{number of vertices} \end{aligned}$$

It is straightforward to see that the superficial degree of divergence is given by

$$D = 4 - B \quad (17)$$



Fig 5, counter terms for 2-point function

It is important to note that D depends only on the number of external lines, B and not on n , the number of vertices. This is a consequence of $\lambda\phi^4$ theory and might not hold for other interactions. From this Eq (17) we see that $D \geq 0$ only for $B = 2, 4$ ($B = \text{even}$ because of the symmetry $\phi \rightarrow -\phi$). In the analysis of divergences, we will use the superficial degree of divergences to construct the counterterms. The reason for this will be explained later.

1. $B = 2, \Rightarrow D = 2$

Being quadratically divergent, the necessary Taylor expansion for the 2-point function is of the form,

$$\Sigma(p^2) = \Sigma(0) + p^2 \Sigma'(0) + \tilde{\Sigma}(p^2)$$

where $\Sigma(0)$ and $\Sigma'(0)$ are divergent and $\tilde{\Sigma}(p^2)$. To cancel these divergences we need to add two counterterms,

$$\frac{1}{2} \Sigma(0) \phi^2 + \frac{1}{2} \Sigma'(0) (\partial_\mu \phi)^2$$

which give the following contributions,

2. $B = 4, \Rightarrow D = 0$

The Taylor expansion is

$$\Gamma^{(4)}(p_i) = \Gamma^{(4)}(0) + \tilde{\Gamma}^{(4)}(p_i)$$

where $\Gamma^{(4)}(0)$ is logarithmically divergent which is to be cancelled by counterterm of the form

$$\frac{i}{4!} \Gamma^{(4)}(0) \phi^4$$

The general counterterm Lagrangian is then

$$\Delta\mathcal{L} = \frac{1}{2} \Sigma(0) \phi^2 + \frac{1}{2} \Sigma'(0) (\partial_\mu \phi)^2 + \frac{i}{4!} \Gamma^{(4)}(0) \phi^4$$

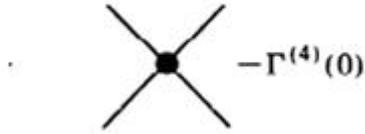


Fig 6 counterterms for 4-point function

which is clearly the same as Eq(16) with the identification

$$\begin{aligned}\Sigma'(0) &= (Z_\varphi - 1) \\ \Sigma(0) &= -(Z_\varphi - 1)\mu^2 + \delta\mu^2 \\ \Gamma^{(4)}(0) &= -i\lambda(1 - Z_\lambda)\end{aligned}$$

This illustrates the equivalence of BPH renormalization and conventional renormalization.

More on BPH renormalization

The BPH renormalization scheme looks very simple. It is remarkable that this simple scheme can serve as the basis for setting up a proof for a certain class of field theory. There are many interesting and useful features in BPH which do not show themselves on the first glance and are very useful in the understanding of this renormalization program. We will now discuss some of them.

1. Convergence of Feynman diagrams

In our analysis so far, we have used the superficial degree of divergences D . It is clear that to 1-loop order that superficial degree of divergence is the same as the real degree of divergence. When we go beyond 1-loop it is possible to have an overall $D < 0$ while there are real divergences in the subgraphs. The real convergence of a Feynman graph is governed by Weinberg's theorem ([7]) : The general Feynman integral converges if the superficial degree of divergence of the graph together with the superficial degree of divergence of all subgraphs are negative. To be more explicit, consider a Feynman graph with n external lines and l loops. Introduce a cutoff Λ in the momentum integration to estimate the order of divergence,

$$\Gamma^{(n)}(p_1, \dots, p_{n-1}) = \int_0^\Lambda d^4q_1 \cdots d^4q_l I(p_1, \dots, p_{n-1}; q_1, \dots, q_l)$$

Take a subset $S = \{q'_1, q'_2, \dots, q'_m\}$ of the loop momenta $\{q_1, \dots, q_l\}$ and scale them to infinity and all other momenta fixed. Let $D(S)$ be the

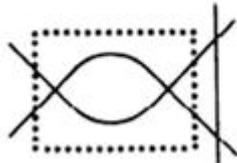


Fig 7 divergence in 6-point function

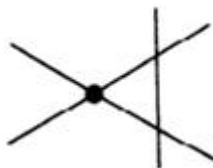


Fig 8 Counterterm for 6-point function

superficial degree of divergence associated with integration over this set, i. e.,

$$\left| \int_0^\Lambda d^4 q'_1 \cdots d^4 q'_m I \right| \leq \Lambda^{D(s)} \{ \ln \Lambda \}$$

where $\{ \ln \Lambda \}$ is some function of $\ln \Lambda$. Then the convergent theorem states that the integral over $\{ q_1, \cdots, q_i \}$ converges if the $D(S)'$ s for all possible choice of S are negative. For example the graph in the following figure

is a 6-point function with $D = -2$. But the integration inside the box with $D = 0$ is logarithmically divergent. However, in the BPH procedure this subdivergence is in fact removed by lower order counter terms as shown below.

2. Classification of divergent graphs

It is useful to distinguish divergent graphs with different topologies in the construction of counterterms.

(a) Primitively divergent graphs

A primitively divergent graph has a nonnegative overall superficial degree of divergence but is convergent for all subintegrations. Thus these are diagrams in which the only divergences is caused by all of the loop momenta growing large together. This means that when we differentiate with respect to external momenta at

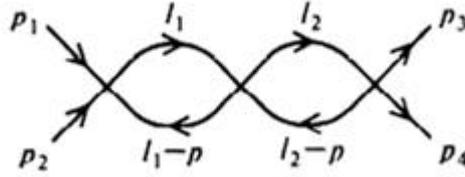


Fig 9 two-loop disjoint divergence

least one of the internal loop momenta will have more power in the denominator and will improve the convergence of the diagram. It is then clear that all the divergences can be isolated in the first few terms of the Taylor expansion.

(b) Disjointed divergent graphs

Here the divergent subgraphs are disjoint. For illustration, consider the 2-loop graph given below,

It is clear that differentiating with respect to the external momentum will improve only one of the loop integration but not both. As a result, not all divergences in this diagram can be removed by subtracting out the first few terms in the Taylor expansion around external momenta. However, the lower order counter terms in the BPH scheme will come in to save the day. The Feynman integral is written as

$$\Gamma_a^{(4)}(p) \propto \lambda^3 [\Gamma(p)]^2$$

with

$$\Gamma(p) = \frac{1}{2} \int d^4l \frac{1}{l^2 - \mu^2 + i\varepsilon} \frac{1}{[(l-p)^2 - \mu^2 + i\varepsilon]}$$

and $p = p_1 + p_2$. Since $\Gamma(p)$ is logarithmic divergent, $\Gamma_a^{(4)}(p)$ cannot be made convergent no matter how many derivatives act on it, even though the overall superficial degree of divergence is zero. However, we have the lower order counterterm $-\lambda^2\Gamma(0)$ corresponding to the subtraction introduced at the 1-loop level. This generates the additional contributions given in the following diagrams,

which are proportional to $-\lambda^3\Gamma(0)\Gamma(p)$. Adding these 3 contributions, we get

$$\begin{aligned} & \lambda^3 [\Gamma(p)]^2 - 2\lambda^3\Gamma(0)\Gamma(p) \\ & = \lambda^3 [\Gamma(p) - \Gamma(0)]^2 - \lambda^3 [\Gamma(0)]^2 \end{aligned}$$



Fig 10 two-loop graphs with counterterms

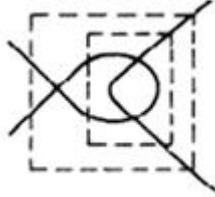


Fig 11 Nested divergences

Since the combination in the first $[\dots]$ is finite, the divergence in the last term can be removed by one differentiation. Here we see that with the inclusion of lower order counterterms, the divergences take the form of polynomials in external momenta. Thus for graphs with disjoint divergences we need to include the lower order counter terms to remove the divergences by subtractions in Taylor expansion.

(c) Nested divergent graphs

In this case one of a pair of divergent 1PI is entirely contained within the other as shown in the following diagram,

After the subgraph divergence is removed by diagrams with lower order counterterms, the overall divergences is then renormalized by a λ^3 counter terms as shown below,

Again diagrams with lower-order counterterm insertions must be included in order to aggregate the divergences into the form of polynomial in external momenta.

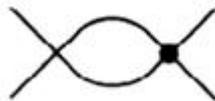


Fig 12 lower order counterterm

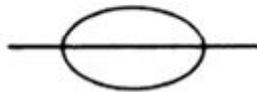


Fig 13 Overlapping divergences

(d) Overlapping divergent graphs

These diagrams are those divergences which are neither nested nor disjointed. These are most difficult to analyze. An example of this is shown below,

The study of how to disentangle these overlapping divergences is beyond the scope of this simple introduction and we refer interested readers to the literature ([3].[4]).

From these discussion, it is clear that BPH renormalization scheme is quite useful in organizing the higher order divergences in a more systematic way for the removing of divergences by constructing the counterterms.

The general analysis of the renormalization program has been carried out by Bogoliubov, Parasiuk, Hepp ([3]). The result is known as BPH theorem, which states that for a general renormalizable field theory, to any order in perturbation theory, all divergences are removed by the counterterms corresponding to superficially divergent amplitudes.

3 Power counting and Renormalizability

We now discuss the problem of renormalization for more general interactions. It is clear that it is advantageous to use the BPH scheme in this discussion.

3.1 Theories with fermions and scalar fields

We first study the simple case with fermion ψ and scalar field ϕ . Write the Lagrangian density as

$$\mathcal{L} = \mathcal{L}_0 + \sum_i \mathcal{L}_i$$

where \mathcal{L}_0 is the free Lagrangian quadratic in the fields and \mathcal{L}_i are the interaction terms e.g.

$$\mathcal{L}_i = g_1 \bar{\psi} \gamma^\mu \psi \partial_\mu \phi, \quad g_2 (\bar{\psi} \psi)^2, \quad g_3 (\bar{\psi} \psi) \phi, \quad \dots$$

Here ψ denotes a fermion field and ϕ a scalar field. Define the following quantities

$$\begin{aligned}
n_i &= \text{number of } i\text{-th type vertices} \\
b_i &= \text{number of scalar lines in } i\text{-th type vertex} \\
f_i &= \text{number of fermion lines in } i\text{-th type vertex} \\
d_i &= \text{number of derivatives in } i\text{-th type of vertex} \\
B &= \text{number of external scalar lines} \\
F &= \text{number of external fermion lines} \\
IB &= \text{number of internal scalar lines} \\
IF &= \text{number of internal fermion lines}
\end{aligned}$$

Counting the scalar and fermion lines, we get

$$B + 2(IB) = \sum_i n_i b_i \quad (18)$$

$$F + 2(IF) = \sum_i n_i f_i \quad (19)$$

Using momentum conservation at each vertex we can compute the number of loop integration L as

$$L = (IB) + (IF) - n + 1, \quad n = \sum_i n_i$$

where the last term is due to the overall momentum conservation which does not contain the loop integrations. The superficial degree of divergence is then given by

$$D = 4L - 2(IB) - (IF) + \sum_i n_i d_i$$

Using the relations given in Eqs(18,19) we get

$$D = 4 - B - \frac{3}{2}F + \sum_i n_i \delta_i \quad (20)$$

where

$$\delta_i = b_i + \frac{3}{2}f_i + d_i - 4$$

is called the *index of divergence* of the interaction. Using the fact that Lagrangian density \mathcal{L} has dimension 4 and scalar field, fermion field and the derivative have dimensions, 1, $\frac{3}{2}$, and 1 respectively, we get for the dimension of the coupling constant g_i as

$$\dim(g_i) = 4 - b_i - \frac{3}{2}f_i - d_i = -\delta_i$$

We distinguish 3 different situations;

1. $\delta_i < 0$

In this case, D decreases with the number of i -th type of vertices and the interaction is called *super – renormalizable interaction*. The divergences occur only in some lower order diagrams. There is only one type of theory in this category, namely ϕ^3 interaction.

2. $\delta_i = 0$

Here D is independent of the number of i -th type of vertices and interactions are called *renormalizable interactions*. The divergence are present in all higher-order diagrams of a finite number of Green's functions. Interactions in this category are of the form, $g\phi^4, f(\bar{\psi}\psi)\phi$.

3. $\delta_i > 0$

Then D increases with the number of the i -th type of vertices and all Green's functions are divergent for large enough n_i . These are called *non – renormalizable interactions*. There are plenty of examples in this category, $g_1\bar{\psi}\gamma^\mu\psi\partial_\mu\phi, g_2(\bar{\psi}\psi)^2, g_3\phi^5, \dots$

The index of divergence δ_i can be related to the operator's *canonical dimension* which is defined in terms of the high energy behavior in the free field theory. More specifically, for any operator A , we write the 2-point function as

$$D_A(p^2) = \int d^4x e^{-ip \cdot x} \langle 0 | T(A(x)A(0)) | 0 \rangle$$

If the asymptotic behavior is of the form,

$$D_A(p^2) \longrightarrow (p^2)^{-\omega_A/2}, \quad \text{as } p^2 \longrightarrow \infty$$

then the canonical dimension is defined as

$$d(A) = (4 - \omega_A) / 2$$

Thus for the case of fermion and scalar fields we have,

$$\begin{aligned} d(\phi) &= 1, & d(\partial^n \phi) &= 1 + n \\ d(\psi) &= \frac{3}{2}, & d(\partial^n \psi) &= \frac{3}{2} + n \end{aligned}$$

Note that in these simple cases, these values are the same as those obtained in the dimensional analysis in the classical theory and sometimes they are also called the naive dimensions. As we will see later for the vector field, the canonical dimension is not necessarily the same as the naive dimension.

For composite operators that are polynomials in the scalar or fermion fields it is difficult to know their asymptotic behavior. So we define their canonical dimensions as the algebraic sum of their constituent fields. For example,

$$d(\phi^2) = 2, \quad d(\bar{\psi}\psi) = 3$$

For general composite operators that show up in the those interaction described before, we have,

$$d(\mathcal{L}_i) = b_i + \frac{3}{2}f_i + d_i$$

and it is related to the index of divergence as

$$\delta_i = d(\mathcal{L}_i) - 4$$

We see that a dimension 4 interaction is renormalizable and greater than 4 is non-renormalizable.

Counter terms

Recall that we add counterterms to cancel all the divergences in Green's functions with superficial degree of divergences $D \geq 0$. For convenience we use the Taylor expansion around zero external momenta $p_i = 0$. It is easy to see that a general diagram with $D \geq 0$, counter terms will be of the form

$$O_{ct} = (\partial_\mu)^\alpha (\psi)^F (\phi)^B, \quad \alpha = 1, 2, \dots, D$$

and the canonical dimension is

$$d_{ct} = \frac{3}{2}F + B + \alpha$$

The index of divergence of the counterterms is

$$\delta_{ct} = d_{ct} - 4$$

Using the relation in Eq (20) we can write this as

$$\delta_{ct} = (\alpha - D) + \sum_i n_i \delta_i$$

Since $\alpha \leq D$, we have the result

$$\delta_{ct} \leq \sum_i n_i \delta_i$$

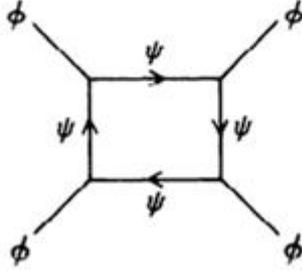


Fig 14 Box diagram for Yukawa coupling

Thus, the counterterms induced by a Feynman diagrams have indices of divergences less or equal to the sum of the indices of divergences of all interactions δ_i in the diagram.

We then get the general result that the renormalizable interactions which have $\delta_i = 0$ will generate counterterms with $\delta_{ct} \leq 0$. Thus if all the $\delta_i \leq 0$ terms are present in the original Lagrangian, so that the counter terms have the same structure as the interactions in the original Lagrangian, they may be considered as redefining parameters like masses and coupling constants in the theory. On the other hand non-renormalizable interactions which have $\delta_i > 0$ will generate counterterms with arbitrary large δ_{ct} in sufficiently high orders and clearly cannot be absorbed into the original Lagrangian by a redefinition of parameters δ_{ct} . Thus non-renormalizable theories will not necessarily be infinite; however the infinite number of counterterms associated with a non-renormalizable interaction will make it lack in predictive power and hence be unattractive, in the framework of perturbation theory.

We will adopt a more restricted definition of renormalizability: a Lagrangian is said to be renormalizable by power counting if all the counterterms induced by the renormalization procedure can be absorbed by redefinitions of parameters in the Lagrangian. With this definition the theory with Yukawa interaction $\bar{\psi}\gamma_5\psi\phi$ by itself, is not renormalizable even though the coupling constant is dimensionless. This is because the 1-loop diagram shown below

is logarithmically divergent and needs a counter term of the form ϕ^4 which is not present in the original Lagrangian. Thus Yukawa interaction with additional ϕ^4 interaction is renormalizable.

Theories with vector fields

Here we distinguish massless from massive vector fields because their asymptotic behaviors for the free field propagators are very different.

1. Massless vector field

Massless vector field is usually associated with local gauge invariance as in the case of QED. The asymptotic behavior of free field propagator for such vector field is very similar to that of scalar field. For example, in the Feynman gauge we have

$$\Delta_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} \longrightarrow O(k^{-2}), \quad \text{for large } k^2$$

which has the same asymptotic behavior as that of scalar field. Thus the power counting for theories with massless vector field interacting with fermions and scalar fields is the same as before. The renormalizable interactions in this category are of the type,

$$\bar{\psi}\gamma_\mu\psi A^\mu, \quad \phi^2 A_\mu A^\mu, \quad (\partial_\mu\phi)\phi A^\mu$$

Here A^μ is a massless vector field and ψ a fermion field.

2. Massive vector field

Here the free Lagrangian is of the form,

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \frac{1}{2}M_V^2 V_\mu^2$$

where V_μ is a massive vector field and M_V is the mass of the vector field. The propagator in momentum space is of the form,

$$D_{\mu\nu}(k) = \frac{-i(g_{\mu\nu} - k_\mu k_\nu / M_V^2)}{k^2 - M_V^2 + i\varepsilon} \longrightarrow O(1), \quad \text{as } k \rightarrow \infty \quad (21)$$

This implies that canonical dimension of massive vector field is two rather than one. The power counting is now modified with superficial degree of divergence given by

$$D = 4 - B - \frac{3}{2}F - V + \sum_i n_i (\Delta_i - 4)$$

with

$$\Delta_i = b_i + \frac{3}{2}f_i + 2v_i + d_i$$

Here V is the number of external vector lines, v_i is the number of vector fields in the i th type of vertex and Δ_i is the canonical dimension of the interaction term in \mathcal{L} . From the formula for Δ_i we see that the only renormalizable interaction involving massive vector field, $\Delta_i \leq 4$, is of the form, $\phi^2 A_\mu$ and is not Lorentz invariant. Thus there is no non-trivial interaction of the massive vector field which is renormalizable. However, two important exceptions should be noted;

- (a) In a gauge theory with spontaneous symmetry breaking, the gauge boson will acquire mass in such a way to preserve the renormalizability of the theory ([8]).
- (b) A theory with a neutral massive vector boson coupled to a conserved current is also renormalizable. Heuristically, we can understand this as follows. The propagator in Eq(21) always appears between conserved currents $J^\mu(k)$ and $J^\nu(k)$ and the $k_\mu k_\nu / M_V^2$ term will not contribute because of current conservation, $k^\mu J_\mu(k) = 0$ or in the coordinate space $\partial^\mu J_\mu(x) = 0$. Then the power counting is essentially the same as for the massless vector field case.

3.2 Renormalization of Composite Operators

In some cases, we need to consider the Green's function of composite operator, an operator with more than one fields at the same space time. Consider a simple composite operator of the form $\Omega(x) = \frac{1}{2}\phi^2(x)$ in $\lambda\phi^4$ theory. Green's function with one insertion of Ω is of the form,

$$G_\Omega^{(n)}(x; x_1, x_2, x_3, \dots, x_n) = \left\langle 0 | T \left(\frac{1}{2} \phi^2(x) \phi(x_1) \phi(x_2) \dots \phi(x_n) \right) | 0 \right\rangle$$

In momentum space we have

$$\begin{aligned} & (2\pi)^4 \delta^4(p + p_1 + p_2 + \dots + p_n) G_{\phi^2}^{(n)}(p; p_1, p_2, p_3, \dots, p_n) \\ &= \int d^4x e^{-ipx} \int \prod_{i=1}^n d^4x_i e^{-ip_i x_i} G_\Omega^{(n)}(x; x_1, x_2, x_3, \dots, x_n) \end{aligned}$$

In perturbation theory, we can use Wick's theorem ([9]) to work out these Green's functions in terms of Feynman diagram.

Example, to lowest order in λ the 2-point function with one composite operator $\Omega(x) = \frac{1}{2}\phi^2(x)$ is, after using the Wick's theorem,

$$G_{\phi^2}^{(2)}(x; x_1, x_2) = \frac{1}{2} \langle 0 | T \{ \phi^2(x) \phi(x_1) \phi(x_2) \} | 0 \rangle = i\Delta(x - x_1) i\Delta(x - x_2)$$

or in momentum space

$$G_{\phi^2}^{(2)}(p; p_1, p_2) = i\Delta(p_1) i\Delta(p + p_1)$$

If we truncate the external propagators, we get

$$\Gamma_{\phi^2}^{(2)}(p, p_1, -p_1 - p) = 1$$

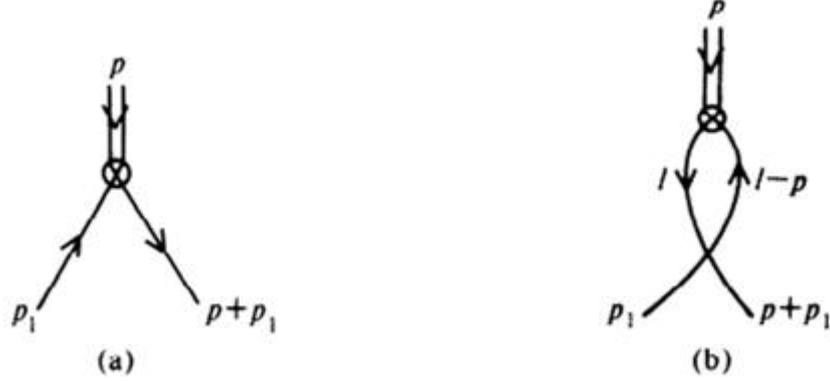


Fig 15 Graphs for composite operator

To first order in λ , we have

$$\begin{aligned}
 G_{\phi^2}^{(2)}(x, x_1, x_2) &= \int \left\langle 0 | T \left\{ \frac{1}{2} \phi^2(x) \phi(x_1) \phi(x_2) \frac{(-i\lambda)}{4!} \phi^4(y) \right\} | 0 \right\rangle d^4 y \\
 &= \int d^4 y \frac{-i\lambda}{2} [i\Delta(x-y)]^2 i\Delta(x_1-y) i\Delta(x_2-y)
 \end{aligned}$$

The amputated 1PI momentum space Green's function is

$$\Gamma_{\phi^2}^{(2)}(p; p_1, -p - p_1) = \frac{-i\lambda}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - \mu^2 + i\epsilon} \frac{i}{(l-p)^2 - \mu^2 + i\epsilon}$$

To calculate this type of Green's functions systematically, we can add a term $\chi(x)\Omega(x)$ to \mathcal{L}

$$\mathcal{L}[\chi] = \mathcal{L}[0] + \chi(x)\Omega(x)$$

where $\chi(x)$ is a c-number source function. We can construct the generating functional $W[\chi]$ in the presence of this external source. We obtain the connected Green's function by differentiating $\ln W[\chi]$ with respect to χ and then setting χ to zero.

Renormalization of composite operators

Superficial degrees of divergence for Green's function with one composite operator is,

$$D_\Omega = D + \delta_\Omega = D + (d_\Omega - 4)$$

where d_Ω is the canonical dimension of Ω . For the case of $\Omega(x) = \frac{1}{2}\phi^2(x)$, $d_{\phi^2} = 2$ and $D_{\phi^2} = 2 - n \Rightarrow$ only $\Gamma_{\phi^2}^{(2)}$ is divergent. Taylor expansion takes the form,

$$\Gamma_{\phi^2}^{(2)}(p; p_1) = \Gamma_{\phi^2}^{(2)}(0, 0) + \Gamma_{\phi^2 R}^{(2)}(p, p_1)$$

We can combine the counter term

$$\frac{-i}{2}\Gamma^{(2)}\phi^2(0,0)\chi(x)\phi^2(x)$$

with the original term to write

$$\frac{-i}{2}\chi\phi - \frac{i}{2}\Gamma_{\phi^2}^2(0,0)\chi\phi^2 = -\frac{i}{2}Z_{\phi^2}\chi\phi^2$$

In general, we need to insert a counterterm $\Delta\Omega$ into the original addition

$$L \rightarrow L + \chi(\Omega + \Delta\Omega)$$

If $\Delta\Omega = C\Omega$, as in the case of $\Omega = \frac{1}{2}\phi^2$, we have

$$L[\chi] = L[0] + \chi Z_{\Omega}\Omega = L[0] + \chi\Omega_0$$

with

$$\Omega_0 = Z_{\Omega}\Omega = (1 + C)\Omega$$

Such composite operators are said to be multiplicative renormalizable and Green's functions of unrenormalized operator Ω_0 is related to that of renormalized operator Ω by

$$\begin{aligned} G_{\Omega_0}^{(n)}(x; x_1, x_2, \dots, x_n) &= \langle 0|T\{\Omega_0(x)\phi(x_1)\phi(x_2)\dots\phi(x_n)\}|0\rangle \\ &= Z_{\Omega}Z_{\phi}^{n/2}G_{IR}^{(n)}(x; x_1, \dots, x_n) \end{aligned} \quad (22)$$

For more general cases, $\Delta\Omega \neq c\Omega$ and the renormalization of a composite operator may require counterterms proportional to other composite operators. Example: Consider 2 composite operators A and B . Denote the counterterms by ΔA and ΔB . Including the counter terms we can write,

$$L[\chi] = L[0] + \chi_A(A + \Delta A) + \chi_B(B + \Delta B)$$

Very often with counterterms ΔA and ΔB are linear combinations of A and B

$$\begin{aligned} \Delta A &= C_{AA}A + C_{AB}B \\ \Delta B &= C_{BA}A + C_{BB}B \end{aligned}$$

We can write

$$L[\chi] = L[0] + (\chi_A \ \chi_B) \{C\} \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{where } \{C\} = \begin{pmatrix} 1 + C_{AA} & C_{AB} \\ C_{BA} & 1 + C_{BB} \end{pmatrix}$$

Diagonalize $\{C\}$ by bi-unitary transformation

$$U\{C\}V^+ = \begin{pmatrix} Z_{A'} & 0 \\ 0 & Z_{B'} \end{pmatrix}$$

Then

$$L[\chi] = L[0] + Z_{A'}\chi_{A'}A' + Z_{B'}\chi_{B'}B'$$

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = V \begin{pmatrix} A \\ B \end{pmatrix} \quad (\chi_{A'} \ \chi_{B'}) = (\chi_A \ \chi_B)U$$

and A', B' are multiplicatively renormalizable.

3.3 Symmetry and Renormalization

For a theory with global symmetry, we require that the counter terms should also respect the symmetry. For example, consider the Lagrangian given by

$$\mathcal{L} = \frac{1}{2} [(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2] - \frac{\mu^2}{2} (\phi_1^2 + \phi_2^2) - \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2 \quad (23)$$

This Lagrangian has the $O(2)$ symmetry given below

$$\phi_1 \rightarrow \phi'_1 = \cos\theta\phi_1 + \sin\theta\phi_2$$

$$\phi_2 \rightarrow \phi'_2 = -\sin\theta\phi_1 + \cos\theta\phi_2$$

The counter terms for this theory should have the same symmetry. For example the mass counter term should be of the form

$$\delta\mu^2 (\phi_1^2 + \phi_2^2)$$

i.e. the coefficient of ϕ_1^2 counter term should be the same as ϕ_2^2 term. Then the only other possible counter terms are of the form,

$$(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2, \quad (\phi_1^2 + \phi_2^2)^2$$

1. Broken symmetry and renormalization

For the case the symmetry is slightly broken an interesting feature occurs. We will illustrate this with a simple case where the symmetry breaking is of the form,

$$\mathcal{L}_{SB} = c(\phi_1^2 - \phi_2^2)$$

Since the index of divergence for \mathcal{L}_{SB} is $\delta_{SB} = -2$, the superficial degree of divergence for graphs containing \mathcal{L}_{SB} is

$$D_{SB} = 4 - B_1 - B_2 - 2n_{SB}$$

where B_1, B_2 are number of external ϕ_1, ϕ_2 lines and n_{SB} is the number of times \mathcal{L}_{SB} appears in the graph. For the case $n_{SB} = 1$, we have

$$D_{SB} = 2 - B_1 - B_2$$

This means that $D_{SB} \geq 0$ only for $B_1 = 2, B_2 = 0$, or $B_1 = 0, B_2 = 2$ and the counter terms we need are ϕ_1^2 , and ϕ_2^2 . The combination $\phi_1^2 + \phi_2^2$ can be absorbed in the mass counter term while the other combination $\phi_1^2 - \phi_2^2$ can be absorbed into \mathcal{L}_{SB} . This shows the when the symmetry is broken, the counterterms we need will have the property that,

$$\delta_{CT} \leq \delta_{SB}$$

Or in terms of operator dimension

$$\dim(\mathcal{L}_{CT}) \leq \dim(\mathcal{L}_{SB})$$

Thus when $\dim(\mathcal{L}_{SB}) \leq 3$, the dimension of counter terms cannot be 4. This situation is usually referred to as soft breaking of the symmetry. This is known as the *Szymanzik theorem* ([10]). Note that for the soft breaking the coupling constant g_{SB} will have positive dimension of mass and will be negligible when energies become much larger than g_{SB} . In other words, the symmetry will be restored at high energies.

2. Ward Identity ([11])

In case of global symmetry, we also have some useful relation for composite operator like the current operator which generates the symmetry. We will give a simple illustration of this feature. The Lagrangian given in Eq (23) can be rewritten as

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial_\mu \phi - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

where

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2)$$

The symmetry transformation is then

$$\phi \rightarrow \phi' = e^{i\theta} \phi$$

This will give rise, through Noether's theorem, the current of the form,

$$J_\mu = i [(\partial_\mu \phi^\dagger) \phi - (\partial_\mu \phi) \phi^\dagger]$$

is conserved,

$$\partial^\mu J_\mu = 0$$

From the canonical commutation relation,

$$[\partial_0 \phi^\dagger(\vec{x}, t), \phi(\vec{x}', t)] = -i \delta^3(\vec{x} - \vec{x}')$$

we can derive,

$$[J_0(\vec{x}, t), \phi(\vec{x}', t)] = \delta^3(\vec{x} - \vec{x}') \phi(\vec{x}', t) \quad (24)$$

$$[J_0(\vec{x}, t), \phi^\dagger(\vec{x}', t)] = -\delta^3(\vec{x} - \vec{x}') \phi^\dagger(\vec{x}', t) \quad (25)$$

Now consider the Green's function of the form,

$$G_\mu(p, q) = \int d^4x d^4y e^{-iq \cdot x - ip \cdot y} \langle 0 | T (J_\mu(x) \phi(y) \phi^\dagger(0)) | 0 \rangle$$

Multiply q^μ into this Green's function,

$$\begin{aligned} q^\mu G_\mu(p, q) &= -i \int d^4x d^4y e^{-iq \cdot x - ip \cdot y} \partial_x^\mu \langle 0 | T (J_\mu(x) \phi(y) \phi^\dagger(0)) | 0 \rangle \\ &= -i \int d^4x e^{-i(q+p) \cdot x} \langle 0 | T (\phi(x) \phi^\dagger(0)) | 0 \rangle \\ &\quad + i \int d^4x e^{-ip \cdot x} \langle 0 | T (\phi(x) \phi^\dagger(0)) | 0 \rangle \end{aligned}$$

where we have used the current conservation and commutators in Eqs (24,25). The right-hand side here is just the propagator for the scalar field,

$$\Delta(p) = \int d^4x e^{-ip \cdot x} \langle 0 | T (\phi(x) \phi^\dagger(0)) | 0 \rangle$$

and we get

$$-iq^\mu G_\mu(p, q) = \Delta(p+q) - \Delta(p) \quad (26)$$

This is example of Ward identity ([11]).

This relation is derived in terms of unrenormalized fields which satisfy the canonical commutation relation. In terms of renormalized quantities,

$$G_\mu^R(p, q) = Z_\phi^{-1} Z_J^{-1} G_\mu(p, q), \quad \Delta^R(p) = Z_\phi^{-1} \Delta(p)$$

the Ward identity in Eq (26) becomes

$$-iZ_J q^\mu G_\mu^R(p, q) = \Delta^R(p + q) - \Delta^R(p)$$

Since the right-hand side is cutoff independent, Z_J on the left-hand side is also cutoff independent, and we do not need any counter terms to renormalize $J_\mu(x)$. In other words, the conserved current $J_\mu(x)$ is not renormalized as composite operator, i.e. $Z_J = 1$. Thus the relation for the renormalized quantities takes the simple form,

$$-iq^\mu G_\mu^R(p, q) = \Delta^R(p + q) - \Delta^R(p)$$

Such a non-renormalization result holds for many conserved quantities.

I would like to thank Professor Yungui Gong for hospitality during my visit to Chongqing Univeristy of Posts and Telecommunications where parts of this manuscript is written.

References

- [1] R. P. Feynman, Phys. Rev. **74**, 939, 1430 (1948), J. Schwinger , Phys. Rev. **73**, 416 (1948), **75**, 898 (1949), S. Tomonaga, Phys. Rev. **74**, 224 (1948), F. J. Dyson, Phys. Rev. **75**, 486 (1949).
- [2] C. Itzykson, and J.-B. Zuber, "Quantum Field Theory", McGraw-Hill, New York, (1980). N. N. Bogoliubov and D. V. Shirkov, "Introduction to Theories of Quantized Fields" (3rd edition) Wiley-Interscience, New York, (1980). M. E. Peskin and D. Schroeder, "An Introduction to Quantum Field Theory", Addison-Wiley, New York, 1995. S. Weinberg, " The Quantum Theory of Fields" Vol, 1, 2, and 3, Cambridge University Press, Cambridge (1995).
- [3] N. N. Bogoliubov and O. S. Parasiuk, Acta. Math. **97**, 227 (1957), K. Hepp, Comm. Math. Phys. **2**, 301, (1966), W. Zimmermann, In "Lectures on Elementary Particles and Quantum Field Theory", Proc. 1970 Brandeis Summer Institute (ed. S. Deser et al) MIT Press, Cambridge, Mass. (1970).
- [4] W. Zimmermann, in "Lectures on elementary particle and quantum field theory" Pro. 1970 Brandies Summer Institute (ed S. Deser) MIT Press Cambridge, Massachusetts.

- [5] G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 189, (1972), C. G. Bollini and J. J. Giambiagi, Phys. Lett. **40B**, 566, (1972), J. F. Ashmore, Nuovo Cimento Lett. **4**, 289, (1972), G. M. Cicuta and Mortaldi, Nuovo Cimento Lett. **4**, 329, (1972).
- [6] C. G. Callan, Phys. Rev. **D2**, 1541, (1970), K. Szymanzik, Comm. Math. Phys. **18**, 227, (1970).
- [7] S. Weinberg, Phys. Rev. **118**, 838, (1960).
- [8] G. 't Hooft, Nucl. Phys. **B35**, 173, (1971).
- [9] G. C. Wick, Phys. Rev. **80**, 268 (1950).
- [10] K. Szymanzik, in Coral Gables Conf. on Fundamental interactions at high energies II, ed. A. Perlmutter, G. J. Iverson and R.M. Williams (Gordon and Breach, New York, (1970).
- [11] J. C. Ward, Phys. Rev. **78**, 1824, (1950).