Collective Motion in a System of Brownian Agents

Alexandro Heiblum Robles
Posgrado en Ciencias Físicas
Universidad Nacional Autónoma de México
aheiblum@fisica.unam.mx
Francisco J. Sevilla
Instituto de Física
Universidad Nacional Autónoma de México
fjsevilla@fisica.unam.mx

In order to controvert the common conviction that non-equilibrium features, such as self-propulsion and angular (non-thermal) noise, are essential to the emergence of long-range order (flocking behavior) in a two-dimensional system of self-propelled particles, we build a minimal, Vicsek's inspired [1], model based on non-self-propelled (Brownian) agents subject to thermal noise and argue that such features are not a requirement for the spontaneous emergence of flocking behavior. Instead, we focus on the local alignment interaction between agents as the key element that takes the system out of equilibrium. We found that in the disordered phase, the system can be described by linear response theory where local mass flows cancel on the whole system and simplifying its analysis, while by increasing the rate of alignment or the system's density, fluctuations induced by the thermal noise are diminished leading to a far-from-equilibrium phase with a respective breakdown of linear response theory.

1 Introduction

The study of far from equilibrium systems have attracted the attention in many specialties of science due to its potential applicability that ranges from physics to biology passing through the social sciences. One of the interests among the many different aspects of these systems, of particular relevance in physics, is the possibility of exhibiting *phase transitions* to states that present *collective behavior*. A significant result on this direction, with implications in biological systems, was given by the well known model of Vicsek *et al.* [1] or simply Vicsek's model as is known nowadays, that describes the dynamics of motion of self-propelled agents that interact among them by very simple rules, namely, the alignment of the agents velocity vector along the direction of the mean velocity of their nearby neighbors. As shown by various numerical simulations [2, 3], such a dynamic rule favors the emergence of collective motion at high enough densities distinguished by long-range order. This remarkable characteristic has made the concept of self-propelled particles to be considered as a simple, but important, paradigm that captures the essential ingredients of the collective behavior observed in many systems just like mammalian herds [4], crowds of pedestrians [5, 6], bird flocks [7], fish schools [8], insect swarms, bacterial organisms and many others. The model of Vicsek itself has also been subject of a considerable amount of research on the non-equilibrium aspects of self-propelled particles systems. One of these is related to the nature of the order-disorder phase transition which depends essentially on the type of the stochastic perturbation [9, 10, 3] and many others which presents studies on generalizations an modifications of it [11].

Theoretical approaches, different from the rule-based of Vicsek, that describe collective behavior in systems driven far from equilibrium have also appeared in the literature, see Refs. [12, 13, 14, 15] just to mentions some of them. These different approaches, generally more involved in that interactions are modeled as "forces" that perturb the trajectories of agents, allow for a richer description in that are capable of qualitatively reproducing the spatial-patterns observed in many systems that develop collective behavior [12, 13, 14] such as vortex-like structures. These approaches allows for the inclusion of more detailed interactions and are generally used in explicit applications of biological systems. For instance, recent experimental results suggest that in a more detailed level, different mechanisms, like cannibalism, may drive the system to a collective, although not cooperative, behavior [16]. Another model, that considers the effects of nonlinear noise in a system of self-propelled particles [15], shows the random transit, back and forth, between a state of collective motion and a vortex-like state, arguable this feature qualitatively describe many processes in biological systems.

Besides these important and interesting applications of the framework just described, an important conclusion is derived from the simple rule-model of Vicsek: a phase transition to a state with long-range order in 2-dimensions via the *breaking of a continuous symmetry*, rotational in this case, is possible. Such phase transition is forbidden by the Mermin-Wagner-Hohenberg theorem [17, 18] in a situation of thermodynamical equilibrium, this is the reason why the non-equilibrium features of self-propelled particles, namely that they acquire a finite velocity by themselves [19], are assumed to be the key ingredient for such phase transition to occur [20], however, the exact mechanism that originates the long-range-order phase is still not well understood. On this direction, it has been suggested in Ref. [20] that the Vicsek model can be considered as the non-Hamiltonian version of the XY model (which does not exhibit a long-range phase) in the $v_0 \to 0$ limit, being v_0 the self-propulsive velocity of the agents. The major difference is the off-lattice displacements of the particles in the former, while in the later, the classical spins remain fixed at the points of a lattice.

Thus, in spite of the common consensus that non-equilibrium features, such as self-propulsion and angular (non-thermal) noise, are essential ingredients to the emergence of long-range order collective behavior in two-dimensions, clearly manifested in the vast amount of literature on *flocking behavior* that maintain the self-propulsive nature of the agents, in this paper we present evidence that the key ingredient that lead to a phase with long-range-order is the alignment interaction. We arrive at this conclusion by using an alternative model based on stochastic differential equations with the following assumptions: *i*) the self-propelled feature of the particles is absent and *ii*) fluctuations are due to a thermal bath kept at the temperature T. The model suggested in this paper also offers a starting point for the study of different possible mechanisms for the onset of collective motion.

The paper is organized in the following manner. In section §2 we present a generic continuous-time model in terms of stochastic differential equations. The alignment interaction is implemented in order to capture the essence of the alignment rule of Vicsek's model. In section §3 we discuss the results found with respect to the origin of the non-equilibrium phase with long-range-order. Section §4 embodies our conclusions.

2 The model

The model that meet our purposes consists of N agents on a 2-dimensional square of sidelength L, subjected to periodic boundary conditions immersed in an ideal thermal bath. Each agent is characterized by velocity $\boldsymbol{v}$ and position $\boldsymbol{x}$ and the dynamics of the system is given by the following generic Langevin equations

$$\frac{d\boldsymbol{v}_i}{dt} = -\gamma \boldsymbol{v}_i + \boldsymbol{\xi}_i + \boldsymbol{F}_{i,\mathrm{align}}, \qquad \boldsymbol{v}_i = \frac{d\boldsymbol{x}_i}{dt}, \tag{1}$$

where $\boldsymbol{v}_i$ and $\boldsymbol{x}_i$ denote the velocity and position of the i-th agent, respectively. $\boldsymbol{\xi}_i = (\xi_{x,i}, \xi_{y,i})$ is a two component vector with white noise components, *i.e.* $\langle \xi_{\mu,i}(t) \xi_{\nu,j}(s) \rangle = \delta_{\mu,\nu} \delta_{i,j} 2k_B T \gamma \delta(t-s)$, with T the temperature of the thermal bath, k_B the Boltzmann constant and $\delta(x)$ denotes the Dirac δ-function. The connection between the correlations of the fluctuating "force" $\boldsymbol{\xi}$ and the dissipative coefficient γ, guaranties that the stationary probability distribution of the velocities corresponds exactly to that of thermal equilibrium, *i.e. fluctuation-dissipation theorem* holds. $\boldsymbol{F}_{i,\mathrm{align}}$ denotes the local alignment interaction which is specified in more detail below. In the non-interacting case, *i.e.* $\boldsymbol{F}_{i,\mathrm{align}} = 0$, the N agents systems are in thermal equilibrium with the bath and a well defined temperature can be associated. In addition, our model equations reduce to the standard Langevin description of thermal equilibrium of Brownian motion whose stationary state for the distribution of the single-particle velocities is given by the Maxwell distribution

$$P_{eq}(\boldsymbol{v}) = \frac{1}{(2\pi k_B T/m)} \exp\left\{-\frac{mv^2}{2k_B T}\right\}. \tag{2}$$

Once the alignment interaction is turned on, the most general situation corresponds to that in which the fluctuation-dissipation relation lose its validity, defining a non-equilibrium condition. In this case, the temperature of the bath is not the temperature of the interacting N system, but simply a measure for the fluctuations the bath imparts to the system.

The dispersion $\sqrt{\langle(\boldsymbol{v} - \langle\boldsymbol{v}\rangle)^2\rangle} = \sqrt{2k_B T/m}$ provides a natural scale for the speed of the particles, where $\langle\mathcal{O}\rangle$ denotes the average of $\mathcal{O}$ with the probability distribution given in Eq. (2). This equilibrium characteristic is generally taken into account in standard physical systems, but is usually disregarded within the context of non-equilibrium systems described by equations alike Eqs. (1), as is in the model of "flocking" behavior given in Ref. [16], where fluctuation-dissipation theorem is violated.

The alignment interaction $\boldsymbol{F}_i$ is implemented in order to resemble the original idea of Vicsek *et al.* [1] on the interaction mechanism for collective behavior, however, we avoid its infinite alignment rate[1]. First, an agent will tend to align along the direction $\boldsymbol{f}_i$ given by the average direction of motion of the neighbors in the vicinity of radius R denoted with $\Omega_R(i)$, *i.e.*,

$$\boldsymbol{f}_i = \frac{1}{N_R(i)} \sum_{j\in\Omega_R(i)} \hat{\boldsymbol{v}}_j, \tag{3}$$

where $N_R(i)$ is the number of neighboring agents in $\Omega_R(i)$, see Fig. 1, and $\hat{\boldsymbol{v}}_j = \boldsymbol{v}_j/v_i$ gives the direction of motion of the j-th agent with $v_j = |\boldsymbol{v}_j|$.

The "force" responsible of the alignment of velocities is implemented in such a way that: *i*) is independent of the agents speed (for simplicity), *ii*) is directly proportional to $f_i = |\boldsymbol{f}_i|$ and *iii*) acts orthogonally to the direction of motion $\hat{\boldsymbol{v}}_i$. These features are enclosed in

$$\boldsymbol{F}_{i,\text{align}} = \Gamma\left[\boldsymbol{f}_i - \hat{\boldsymbol{v}}_i\,(\boldsymbol{f}_i \cdot \hat{\boldsymbol{v}}_i)\right], \tag{4}$$

where Γ is the coupling constant with units of *velocity/time* that gives the maximum alignment rate.

interactions We want to remark that the alignment interaction given in Eq. (4) does not affect the magnitude of the particle's velocity. Indeed, when Eq. (4) is split in its components along the direction of motion $\hat{\mathbf{v}}_i$, and along the orthonormal vector $\hat{\boldsymbol{\theta}}_i$, it is clear that

$$\begin{aligned}\mathbf{F}_{i,\text{align}} \cdot \hat{\boldsymbol{v}}_i &= \Gamma\left[\boldsymbol{f}_i - \hat{\boldsymbol{v}}_i\,(\boldsymbol{f}_i \cdot \hat{\boldsymbol{v}}_i)\right] \cdot \hat{\boldsymbol{v}}_i \\ &= 0\end{aligned}$$

[1] In the model of Vicsek *at al.* the agents align to the average direction of motion of the local group just between two successive updating steps.

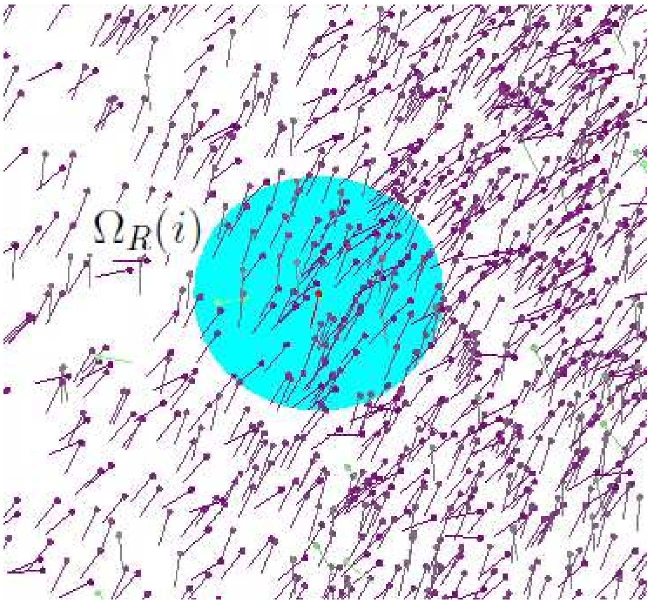

Figure 1: Alignment mechanism. Dots represent the particles positions while lines gives only trace of the direction of motion.

and that

$$\begin{aligned}\mathbf{F}_{i,\mathrm{align}} \cdot \hat{\boldsymbol{\theta}}_i &= \Gamma\, [\boldsymbol{f}_i - \hat{\boldsymbol{v}}_i\, (\boldsymbol{f}_i \cdot \hat{\boldsymbol{v}}_i)] \cdot \hat{\boldsymbol{\theta}}_i \\ &= \Gamma\, \boldsymbol{f}_i \cdot \hat{\boldsymbol{\theta}}_i.\end{aligned}$$

This last expression can be be written in a more suitable form as

$$\mathbf{F}_{i,\mathrm{align}} \cdot \hat{\boldsymbol{\theta}}_i = \Gamma\, f_i\, \sin(\theta_{f_i} - \theta_i),$$

where we have used $\boldsymbol{f}_i = f_i\,(\cos\theta_{f_i}, \sin\theta_{f_i}) = (f_{i,x}, f_{i,y})$, $\theta_{f_i} \equiv \arctan(f_{i,x}/f_{i,y})$ and $\hat{\boldsymbol{\theta}}_i = (-\sin\theta_i, \cos\theta_i)$. Thus, in the absence of dissipation and noise we have

$$\frac{dv_i}{dt} = 0, \tag{5}$$

$$v_i \frac{d\theta_i}{dt} = \frac{\Gamma}{m} f_i \sin(\theta_{f_i} - \theta_i). \tag{6}$$

Note that the alignment mechanism suggested in Eq. (4) leads to equation (6), which is reminiscent of the Kuramoto's mean field equations of synchronization for the velocity's direction of the particles instead of the phase coupled oscillators as in Kuramoto's original model [21]. This aspect has also been considered for the rule-based model of Vicsek [22], where the alignment rule (with static agents as for instance in a 2-dimensional lattice) has been taken to the continuous-time limit and a relation to the Kuramoto model of synchronization has been devised. This contrast with the model here presented, where the number of neighbors in $\Omega_R(i)$ is, due to the non–trivial coupling to the position of the particles, a stochastic variable.

By substituting expression (4) in (1) the equations of the model are explicitly given by

$$\frac{d\boldsymbol{v}_i}{dt} = -\gamma \boldsymbol{v}_i + \Gamma \left[\boldsymbol{f}_i - \hat{\boldsymbol{v}}_i \left(\boldsymbol{f}_i \cdot \hat{\boldsymbol{v}}_i \right) \right] + \boldsymbol{\xi}_i \tag{7}$$

with $\boldsymbol{f}_i$ is defined in (3).

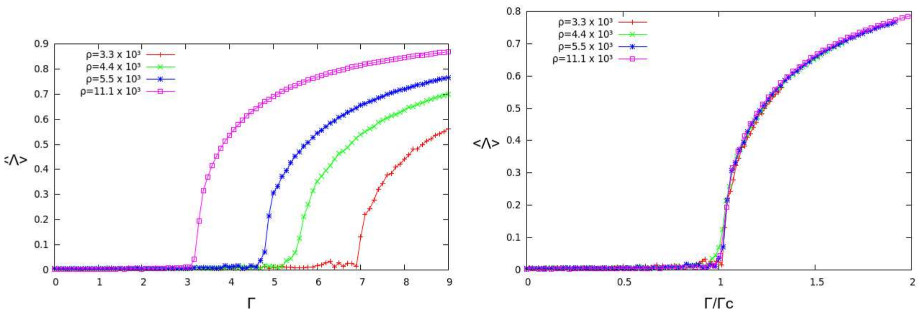

Figure 2: Dependence of the stationary values of the order parameter, $\langle \Lambda \rangle$, on the dimensionless coupling constant $\widetilde{\Gamma}$ for $\rho = 3.3 \times 10^3$, 4.4×10^3, 5.5×10^3 and 11.12×10^3 with $\widetilde{R} = 0.15$. In the left graph it is shown the dependence of the critical value of the coupling constant on the particle density ρ. In the right graph *scaling behavior* is exhibited when $\langle \Lambda \rangle$ is plotted as function of the coupling constant scaled with the critical value.

3 Results and Discussion

In order to analyze the possible solutions of Eqs. (1) we choose γ^{-1} as the time scale of the system. In the interactionless case γ^{-1} corresponds to the relaxation time to the stable equilibrium state defined by zero average velocity. We set $v_0 = \sqrt{2k_BT/m}$ as a scale for velocities and from this we form the length scale $r_0 = \sqrt{2k_BTm/\gamma^2}$. Thus, the number of relevant parameters in our model are: the dimensionless alignment-coupling $\widetilde{\Gamma} = \Gamma/\gamma v_0$, the dimensionless interaction range $\widetilde{R} = R/r_0$ and the dimensionless density of particles $\rho = N/(L/r_0)^2$.

We solve numerically the $4N$ Eqs. (1) by standard methods. Although a fourth order integration algorithm is being prepared, all the results presented in this paper were made with the remarkably fast and simple Euler's method. Using high number of particles, $\sim 10^5$, simulations took, depending on the parameters chosen, an average of $\sim 10^4$ time steps to reach an stationary state, and we perform data collection in about $\sim 10^5$ time steps. Normally this would translate in $24 - 48$ hours of simulation. To characterize the phase transition we define and measure an instantaneous *order parameter*

$$\Lambda(t) = \frac{1}{N} \left| \sum_{j=1}^{N} \hat{\boldsymbol{v}}_i \right|. \tag{8}$$

Our interest is in the stationary value of Λ, denoted with $\langle \Lambda \rangle$ and calculated as

$$\langle \Lambda \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Lambda(t)\, dt. \tag{9}$$

It is clear that in the situation above described of thermal equilibrium ($\Gamma = 0$), any direction of motion of the agents is possible and in the thermodynamic limit, the average of the total velocity of the system vanishes identically giving $\langle \Lambda \rangle = 0$.

With the scales considered above, we find that the system exhibits a *continuous phase transition* from a disordered state, characterized by $\langle \Lambda \rangle = 0$ to an ordered one, $0 < \langle \Lambda \rangle \leq 1$, by varying the coupling constant $\widetilde{\Gamma}$ and fixed values of $\widetilde{R}$ and ρ (see Fig. 2). In the left graph of Fig. 2 it is shown how the critical value $\widetilde{\Gamma}_c$ at which the system attains long-range order is affected by the changing the particle density ρ. In addition, it can be appreciated on the same Fig. 2 that the system exhibits *scaling behavior* when $\langle \Lambda \rangle$ is plotted against $\widetilde{\Gamma}/\widetilde{\Gamma}_c$.

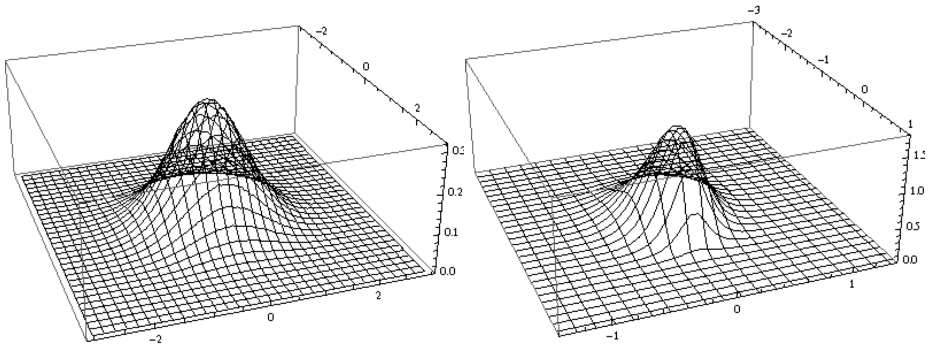

Figure 3: 3D one-body velocity histograms. Left plot is representative of the probability distribution of velocities in the disordered phase ($\tilde{\Gamma} \lesssim 2.2$) that coincide (see next Fig. 4) with the equilibrium probability distribution given by Eq. (2). Right plot gives the corresponding distribution in the ordered phase with $\tilde{\Gamma} = 10.0$.

An interesting result, that might be not so intuitive, is that the single-particle probability distribution of velocities, $P_{st}(\boldsymbol{v})$, for subcritical values of the coupling constant, *i.e.* $0 < \widetilde{\Gamma} \leq \widetilde{\Gamma}_c$, corresponds to the equilibrium one given in Eq. (2). Thus one must expect the system to be close to equilibrium and mean-field theory to be valid in this regime. Furthermore, even when the fluctuation-dissipation relation does not hold a well defined temperature can be assigned to the system. This finding is showed in the left graphs of Fig. 3 and Fig. 4. In Fig. 3 it is shown that the rotationally invariant symmetry of the equilibrium distribution is preserved for the probability distribution of velocities in the disordered phase and that this symmetry is spontaneously broken in the ordered phase as is shown in the right graph on the same figure. In Fig. 4 a quantitative comparison of the velocity distributions with expression (2) is presented for $\widetilde{\Gamma} = 0.1$, 1.0 and 2.2 being $\widetilde{\Gamma}_c \sim 2.2$.in the disordered phase and $\widetilde{\Gamma} = 10.0$ in the ordered one.

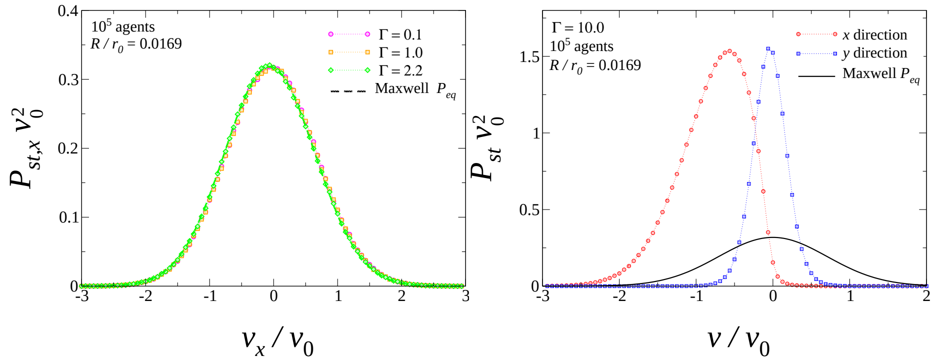

Figure 4: The left graph shows the probability distribution of velocities proyected on the $\boldsymbol{v}_y = 0$ plane for $\widetilde{\Gamma} = 0.1$, 1.0, and 2.2 (disordered phase for $\widetilde{R} = 0.0169$, $\rho = 1.1 \times 10^4$ and $\widetilde{\Gamma}_c \sim 2.2$) that correspond to the rotationally symmetric distribution in the left graph of Fig. 3. Note the agreement with equilibrium distribution of velocities given by (2) (thick-black-dashed line) even at values close to the critical point. The right graph shows the departure from (2) in the order state with $\widetilde{\Gamma} = 10.0$ corresponding to the anisotropic distribution shown in the right grapgh of Fig. 3.

As the coupling parameter Γ is varied from small to large values, the system is self-driven to a far from equilibrium regime when Γ_c is crossed. In this regime the probability distribution of velocities departs from the equilibrium one basically due to the alignment interaction, first the average of the system velocity is finite, thus leading to a state of collective motion and second it appears an anisotropy in the distribution due to the preferred direction of motion spontaneously developed (see right graphs in Fig. 4 and Fig. 3).

The spatial distribution of the local order parameter $\boldsymbol{f}(\boldsymbol{x})$ defined alike to Eq. (3) as

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{1}{N_R(\boldsymbol{x})} \sum_{j \in \Omega_R(\boldsymbol{x})} \hat{\boldsymbol{v}}_j, \tag{10}$$

where now $\boldsymbol{x}$ refers not to the agent position but to a point in the 2-dimensional square of sidelength L, can be related to the local flux of particles $\boldsymbol{j}(\boldsymbol{x})$. One can intuitively expect that the alignment interaction would cause the formation of a local flux whose magnitude would depend on the alignment coupling constant Γ. In the already described disordered phase (subcritical region, $\Gamma < \Gamma_c$) finite local fluxes form, in contrast to the equilibrium case ($\Gamma = 0$) where such fluxes are negligible for large enough systems. On the global, the fluctuations imparted by the bath to the system, makes such local fluxes to fluctuate in practically any direction leading to a vanishing global order parameter. In the ordered state (supercritical region, $\Gamma > \Gamma_c$), such local fluxes dominate over the imparted bath-fluctuations leading to net flux on the global system, *i.e.* to a "flocking". These two situations are explicitly shown in Fig. 5 and Fig. 6, where the 2-dimensional system square has been divided into cells with indexes l_x, l_y and a size of the order of the interaction range. Magnitude and direction of $\boldsymbol{f}(l_x, l_y)$ are explicitly exhibited in Fig. 5 and Fig. 6 respectively.

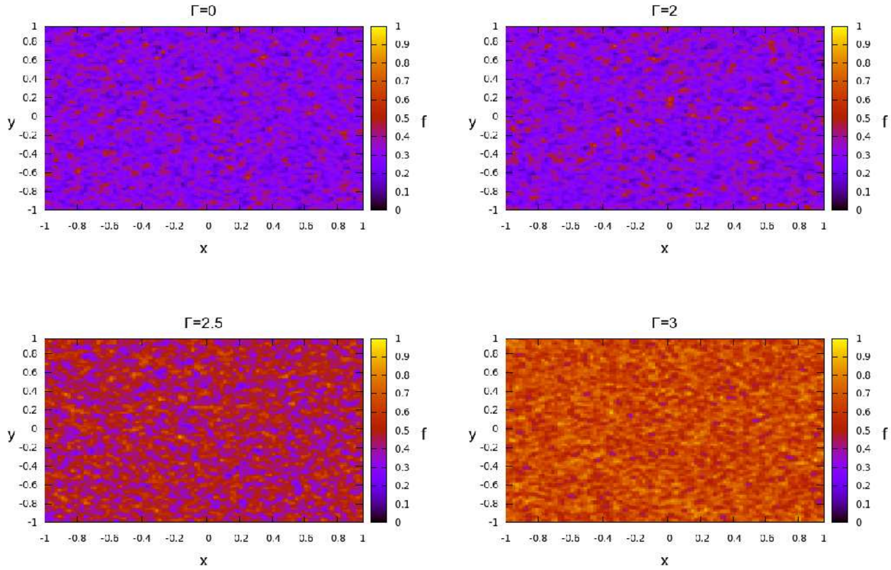

Figure 5: Spatial distribution map of the magnitude $f(\boldsymbol{x})$ of the local order parameter $\boldsymbol{f}(\boldsymbol{x})$, defined as in (3), is shown. The graphs at the top correspond to subcritical values of Γ while the bottom ones for supercritical values. Dark colors correspond to small values of $f(\boldsymbol{x})$ while bright ones to values close to 1.

4 Conclusions and Final Remarks

We have shown that self-propulsion and athermal (non-thermal) noise, two non-equilibrium characteristics in many model that shows flocking behavior, are non necessary for the appearance of states of collective motion. Nor even the weaker, but still out of equilibrium condition presented in Ref. [16] (the fluctuation-dissipation relation between the dissipative and the random forces is clearly violated), is necessary for the appearance of collective states. We achieve this

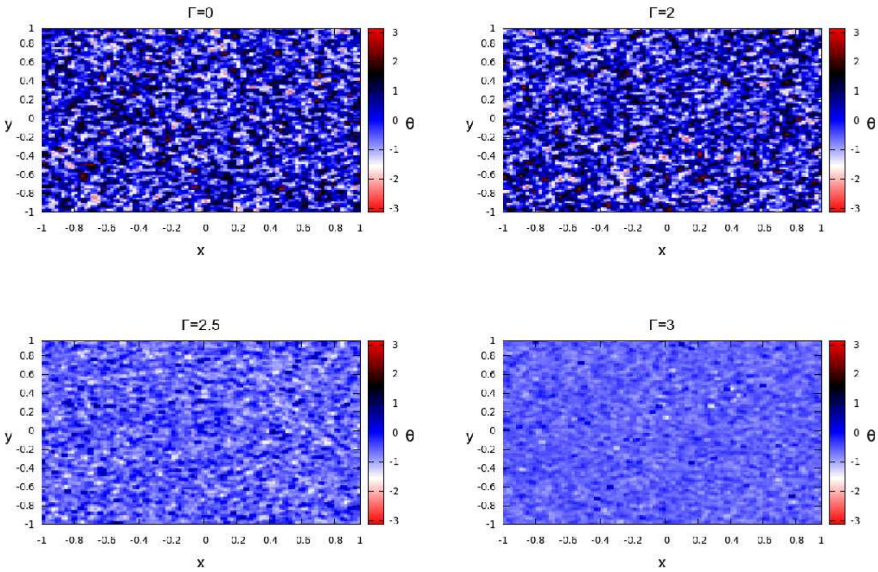

Figure 6: Spatial distribution map of the direction θ of the local order parameter $\boldsymbol{f}(\boldsymbol{x})$ is shown. The graphs at the top correspond to subcritical values of Γ while the bottom ones for supercritical values. Dark colors correspond to small values of $f(\boldsymbol{x}) = |\boldsymbol{f}(\boldsymbol{x})|$ while bright ones to values close to 1.

through the implementation of a model, based on Langevin-like stochastic equations that consider alignment interactions close related to Vicsek's original alignment mechanism. The explicit model interaction for alignment used in this study, resulted to be close connected to the Kuramoto's coupling for synchronization of phase oscillators [21], thus making our model appealing to establish a relation between synchronization and collective-motion behavior.

In addition, it has been shown that in the subcritical regime the probability distribution of single-particle velocities coincides with the equilibrium one (2), and therefore that a well defined temperature can be assigned to the agents collection, this however does not mean the agents are in strict thermodynamical equilibrium. This is actually the case of systems close to thermodynamic equilibrium as is well described by the theory of *linear response* [23]. In fact a two-particle correlations of the velocities study should complement the present analysis, this is the subject of an ongoing work.

Finally, we have found in nature of the interaction, namely, the tendency to move in the same direction of the neighbors, the origin of the possibility of a long-range-order phase in a system in 2 dimensions. A similar conclusion have been suggested in Ref. [24], where extensive Monte Carlo simulations on the original Vicsek model in the $v_0 \to 0$ regime has been performed.

We thank Victor Dossetti for discussions at the initial stage of this project. F.J.S. acknowledges partial support of DGAPA under Grant No. PAPIIT–IN117010.

Bibliography

[1] VICSEK, Tamás, A. CZIRÓK, E. BEN-JACOB, I. COHEN, and O. SHOCHET, "Novel type of phase transition in a system of self-driven particles", *Physical Review Letters* **75** (6) (1995), 1226-1229.

[2] CSAHÓK, Z., A. CZIRÓK, "Hydrodynamics of bacterial motion", *Physica A* **243** (1997), 302.

[3] GRÈGOIRE, Guillaume and H. CHATÉ, "Onset of collective and cohesive motion", *Physical Review Letters* **92** (2004) 025702.

[4] PARRISH, J.K. and W.M. HAMMER (eds.), *Animal Groups in Three Dimensions*, Cambridge University Press (1997).

[5] HELBING, D., I. FARKAS and T. VICSEK, *Nature* **407** (200), 487.

[6] HELBING, D., I. FARKAS and T. VICSEK, *Physical Review Letters* **84** (2000), 1240-1243.

[7] FEARE, C., *Physics Today 60* (2007), 28.

[8] HUBBARD, S., P. BABAK, S. SIGURDSSON and K. MANGUSSON, Ecol. Model. **174**, 359 (2009).

[9] ALDANA, Maximino, Victor DOSSETTI, Christian HUEPE, V.M. KENKRE, Hernán LARRALDE, "Phase transitions in systems of self-propelling agents and related network models", *Physical Review Letters* **98** (9) (2007), 095702.

[10] GRÈGOIRE, Guillaume, H. CHATÉ, and Y. TU, "Moving and staying together without a leader", *Physica D* **181**, 157 (2003); H. CHATÉ, F. GINELLI, G. GRÉGOIRE, and F. RAYNAUD, "Collective motion of self-propelled particles interacting without cohesion", *Physical Review E* **77** (2008), 046113.

[11] CHATÉ, H., F. GINELLI, G. GREGOIRE, F. PERUANI, and F. RAYNAUD, "Modeling Collective Motion: Variations on the Vicsek model", *European Physical Journal B* **64** (2008), 451-456, and references therein.

[12] LEVINE, Herbert, W.-J. RAPPEL, and I. COHEN, "Self-organization in systems of self-propelled particles", *Physical Review E* **63** (2000), 017101.

[13] ERDMANN, Uddo, W. EBELING, and A. S. MIKHAILOV, "Noise-induced transition from translational to rotational motion of swarms", *Physical Review E* **71** (2005), 051904.

[14] D'Orsogna, Maria Rita, Y. L. Chuang, A. L. Bertozzi, and L. S. Chayes, "Self-propelled particles with soft-core interactions: patterns, stability and collapse", *Physical Review Letters* **96** (2006), 104302.

[15] Dossetti, Victor, Francisco J. Sevilla, and V.M. Kenkre, "Phase transitions induced by complex nonlinear noise in a system of self-propelled agents", *Physical Review E* **79** (2009), 051115 .

[16] P. Romanczuk, Ian D. Couzin, and Lutz Schimansky-Geier, "Collective Motion due to Individual Escape and Pursuit Response", *Physical Review Letters* **102** (2009), 010602.

[17] Mermin, N. David, and H. Wagner, "Absence of ferromagnetism or antiferromagnetism in one- or two dimesnional Heisenberg models", *Physical Review Letters* **17** (1966), 1133-1136.

[18] Hohenberg, P.C., "Existence of long-range order in one and two dimensions" Phys. Rev. **158** (1967), 383-386.

[19] Ebeling, Werner, Frank Schweitzer, and Benno Tilch, "Avtive Brownian particles with energy depots modeling animal mobility", *BioSystems* **49** (1999), 17-29.

[20] Czirók, A., Eugene Stanley, and Tamás Vicsek, "Spontaneously ordered motion of self-propelled particles", *Journal of Physics A: Mathematical and General,* **30** (5) (2006), 1375-1385.

[21] Acebrón, Juan A., L. L. Bonilla, Conrad J. Pérez Vicente, Félix Ritort, and Renato Spigler, "The Kuramoto model: A simple paradigm for synchronization phenomena", *Reviews of Modern Physics* **77** (2005), 137-184.

[22] Chepizhko, A.A. and V.L. Kulinskii, "On the relation between Vicsek and Kuramoto models of spontaneous synchronization" *Physica A* **389** (2010), 5347–5352.

[23] Kubo, Ryogo, Morikazu Toda, Natsuki Hashitsume, *Statistical Physics II: Nonequilibrium Statistical Mechanics*, Springer Series in Solid-State Sciences (2003).

[24] Baglietto, Gabriel , Ezequiel V. Albano, "Computer simulations of the collective displacement of self-propelled agents", *Computer Physics Communications* **180** (2009), 527531.