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Universality in the tail of musical note rank distribution

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ABSTRACT

Although power laws have been used to fit rank distributions in many different contexts, they usually fail at the tails. Languages as sequences of symbols have been a popular subject for ranking distributions, and for this purpose, music can be treated as such. Here we show that more than 1800 musical compositions are very well fitted by the first kind two parameter beta distribution, which arises in the ranking of multiplicative stochastic processes. The parameters a and b are obtained for classical, jazz and rock music, revealing interesting features. Specially, we have obtained a clear trend in the values of the parameters for major and minor tonal modes. Finally, we discuss the distribution of notes for each octave and its connection with the ranking of the notes.

1. Introduction

With few exceptions music has been considered as the art which has devoted itself not to the reproduction of natural phenomena, but rather to the expression of the "artist's soul" through sounds. Music is the most "non-material" of the arts. However, musical compositions can be seen as messages written in an alphabet, with different sounds as letters. In most of the musical compositions there are at most 70 different notes (without taking into account the time length of the notes). Such analogy has been the source of many efforts to compare the information content of music with respect to other languages, using tools borrowed from statistics, statistical physics and even fractal geometry. In a text written using a natural language, the elements or "words" can be taken as the different letters, twenty six for English. Other relevant example appears in biology, where the DNA codes the genetic information. In DNA, the "words" are the 61 triplets (codons) without taking into account the STOP word codon. Both natural language texts and DNA sequences present power laws in the observed frequency of a word as a function of its rank (r), where the rank is just the ordinal position of a word, if all words are ordered according to their decreasing frequency. The most frequent word has rank 1, the next most frequent rank 2 and so on. The power law behavior of the ranking is known in languages as the Zipf law [2]. This law is also very common in different fields like in physics, biology, geography, etc. [2]. In physics one can cite the rank distribution of stick-slip events in sheared granular media [3], radionuclides half-life time and nuclides mass number [4]. Other complex systems share the same phenomenology, as networks [5], biological clocks [6] and metabolic networks [7].

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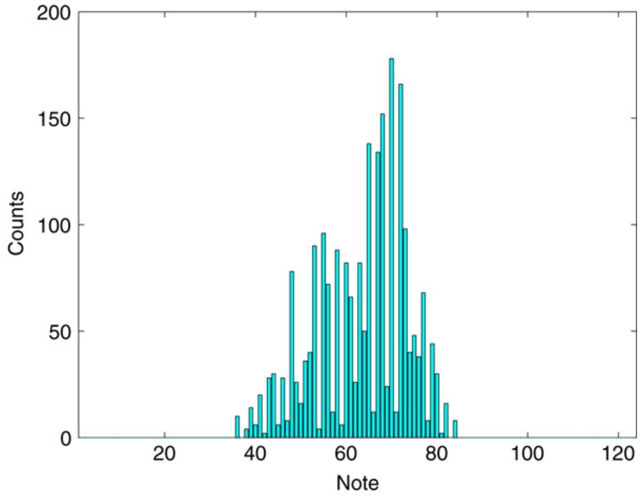

Fig. 1. Number of times that each note is used for Bach's prelude BWV 881 in F minor, taken from the Well Tempered Klavier II.

Many attempts have been made to fit musical compositions with Zipf, Simon and Yule power law formulae with reasonable results [8–11]. However, in most of the fittings presented so far, the power law is not satisfied in all regions, and usually strong deviations are observed at the tails of the ranking [8,9]. Although this is an important feature observed in many different contexts, this observation has not been fully addressed or recognized, even if these events are very important due to their rare appearance. Here we will show that such departures are well fitted by a modified power law, known as the first class two-parameter Beta distribution [1]. In a previous paper [12], we have proved that such law arises in many different systems, like in the ranking of human population, shear-slip events in granular media, in the genome of prokaryotes, in the impact factor of scientific journals [17], etc. Furthermore, we proved that such law arises when multinomial events are ranked in the limit of many random variables. The two-parameter Beta distribution is:

$$f(r) = \frac{K(R+1-r)^b}{r^a}, \tag{1}$$

where $f(r)$ is the frequency of the word r. The parameters a and b are fitted from the data, r is the rank and R is the maximal r. If $f(r)$ is normalized to 1, we have,

$$K \equiv 1 / \sum_{r=1}^{N} \frac{K(R+1-r)^b}{r^a}.$$

For $R \gg 1$, K can be transformed into an integral that yields the Complete Beta Function (not to be confused with the Beta distribution) $K \approx \Gamma(b-a+2)/\Gamma(1-a)\Gamma(1+b)$ where $\Gamma(x)$ is the Gamma function. Notice that Eq. (1) has the virtue of reproducing a power law for intermediate ranges, which means that the Zipf law is valid in the body of the distribution, as is well known from many different studies. Of course that there are many other fitting functions that have been proposed [14–16]. Some of these functions use only one fit parameter, like the Lavalette law, and some others use two, like the Yule–Simon distribution [9]. As discussed in Ref. [12], Eq. (1) uses two parameters, and it is clear that in general it provides a better fit than one-parameter functions at the expense of more parameters. However, Eq. (1) has a theoretical derivation based on the ranking of multinomial events, in which many choices are available in a given step of a tree decision process.

In this article we will show that the tail of the rank-frequency distribution (RFD) for many different musical pieces also follows the same modified beta-like law. As a result, the tail of the RFD is almost universal for music. As we shall see, our results indicate some general trends in the parameters a and b, depending upon the type of music and the tonal scale used by the composer, providing a much more accurate description of music than the Zipf law.

2. Methods and results

The RFD of the musical compositions studied was obtained through their MIDI encoded versions (which is an industry-standard protocol that enables electronic musical instruments, computers and other equipment to communicate, control and synchronize with each other). Each MIDI file can be easily transformed into a standard text file that contain all the information of the composition as a list of all the events in the piece, such as "on" and "off" cues of the notes, speed and volume changes.

These text files are stripped of all other content than the cues of the notes. Each note corresponds to a given pitch. For example, the lowest MIDI note is coded as "0", and corresponds to five octaves below middle C or 8.176 Hz in common Western musical tuning. The highest note is five octaves above the G above middle C or 12,544 Hz, and is designated as MIDI note 127. Thus, the pitch in the MIDI musical alphabet has 128 letters. A typical resulting histogram of the notes is presented in Fig. 1, for Bach's prelude BWV 881 in F minor, taken from the Well Tempered Klavier II. Once the files were translated into

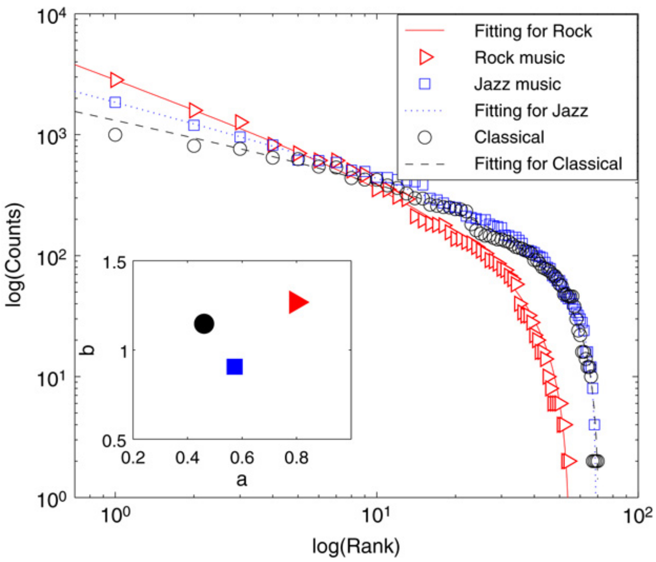

Fig. 2. A log–log plot of the frequency of notes against the rank. The classical piece is Beethoven's Quartet Op. 131 ($a = 0.461$, $b = 1.147$, $r^2 = 0.9968$), the jazz piece is "A good one" by Benny Goodman ($a = 0.573$, $b = 0.905$, $r^2 = 0.9993$) and for rock music, the song "Sweet child of mine" by Guns&Roses ($a = 0.798$, $b = 1.267$, $r^2 = 0.9992$).

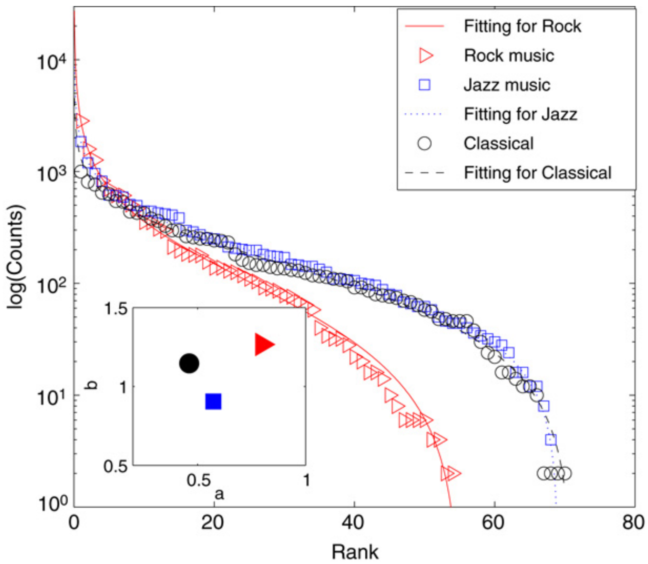

Fig. 3. A semilog plot of the frequency of notes against the rank for the same pieces of the previous figure.

numbers, the notes were ranked according to their frequency. A fit was then performed using Eq. (1), and the parameters a and b were obtained as a product of such fit. All the pieces were analyzed in a quick batch process. The number of pieces using in this study was 941 for classical music, 487 for Jazz and 422 for Rock music.

The spectrum (RFD) and fits of some pieces taken at random for each style analyzed (classical, jazz and rock music) are shown in Figs. 2 and 3. Usually, rank laws are analyzed in log–log plots to compare with a power law, as in Fig. 2. One can clearly see that the power law is only obtained for the most frequent notes, and a tail effect is seen for less frequent notes. In Fig. 3 we present the same set of data as in Fig. 2, but using a semilog plot, since for such kind of graphs the rank axis is not distorted as in the case of the log–log plot. Basically, one can clearly see how Eq. (1) produces an excellent fit, since not only the body of the RFD is well fitted, but also the tail. In fact, the average values of the correlation coefficient ($\mathcal{R}$) is for almost all of the cases larger than 0.99. In Table 1 we present the fitting results done in previous works for several different types of distributions. From the plot of these three particular cases (Beethoven, Benny Goodman and Guns&Roses), one can see that for example, the song by Guns and Roses contains much less harmonic content than the others, since basically notes with a high rank are used. The tail effect in Beethoven and Benny Goodman means that sometimes low rank notes are used, and thus there is more information content. This tail information is encoded in the corresponding values of the parameters

Table 1
Parameters a and b and goodness of fit $\mathcal{R}^2$ of the Beta representation for several symbolic sequences, obtained in previous works

Languages [19]	a	b	$\mathcal{R}^2$
Spanish	0.43	1.31	0.971
French	0.41	1.35	0.967
Latin	0.39	0.86	0.971
English	0.17	1.51	0.964
German	0.39	1.25	0.967
Finnish	0.09	1.41	0.981
Genetic sequences [20]			
Ch. Tracho	0.220	0.501	0.991
E. Coli	0.247	0.503	0.998
Homo Sapiens	0.164	0.365	0.989
Jannasch	0.370	1.243	0.978
Music			
Albeniz (Spanish Suite 5)	1.12	0.39	0.994
Bach (Double Concert BWV 1043)	0.27	1.88	0.988
Beethoven (Quartet Op. 131)	0.20	1.81	0.988
J. Satriani (“Rubina”)	1.78	0.28	0.997
“Dizzy” Gillespie (“Manteca”)	0.79	1.88	0.996

a and b. This can be seen by taking the derivative of $\ln f(r)$,

$$\frac{\mathrm{d}\ln f(r)}{\mathrm{d}r} = -\frac{b}{(R-r+1)} - \frac{a}{r}. \tag{2}$$

For $r \ll R$, corresponding to high rank notes, i.e., the left part of Fig. 2, we have that $(R-r+1) \approx R$ from where it follows that,

$$\frac{\mathrm{d}\ln f(r)}{\mathrm{d}r} \approx -\frac{b}{R(1-(r-1)/R)} - \frac{a}{r} \approx -\frac{a}{r}, \tag{3}$$

so the upward curvature of the tail is basically controlled by the size of parameter a. Furthermore, by taking the integral of Eq. (3) we get,

$$f(r) \approx r^{-a},$$

which allows us to recover the Zipf law. On the other hand, the lowest rank region $r \rightarrow R$, corresponding to the right of Fig. 2, can be approximated as,

$$\frac{\mathrm{d}\ln f(r)}{\mathrm{d}r} \approx -\frac{b}{(R-r)(1-1/(R-r))} - \frac{a}{R} \approx -\frac{b}{(R-r)}. \tag{4}$$

The previous approximation indicates that b controls the statistics of the low rank notes, and it follows a second power law with a cut-off at the maximum range R,

$$f(r) \approx (R-r)^b, \tag{5}$$

Fig. 4 shows a comparison between the fits on a piece by J. Brahms produced with a stretched exponential, a q-exponential [13], a parabolic fractal distribution [15], and the first class two-parameter Beta distribution. It is readily noted that only the TPBD presents a good fitting at both tails of the distribution, owing to its finite domain.

Once the fittings were performed, a pair of a and b parameters was obtained for each of the 1800 pieces. In Fig. 5 we present a plot of the values of b against the parameter a, considering that each piece has a Cartesian coordinate (a, b). Several features are worthwhile mentioning. The first is that we find a few points on the negative region of the a parameter axis. If the parameter a assumes a negative value, Eq. (1) is no longer monotonous and therefore it fails to accurately describe, by definition, a rank-size distribution, even if the fittings are still good. Nevertheless on most cases of negative a, the maximum of the curve described by the fitting is located at values of the rank less than 1, so that the fit is monotonous for admitted values of the rank and is therefore still acceptable. Secondly, rock music shows the smaller dispersion among the data. Jazz music seems to be the greatest dispersion among the values. In Figs. 6 and 7, we show the distribution of a and b for classical, jazz and rock music taken together. For all three cases a is small, and b is centered in a value close to 2. The average value $\langle a \rangle$ for classical music is lower than the average for rock and jazz music.

In terms of the frequency versus rank, that means a more intensive use of less frequent words in the classical music. This is also consistent with the fact that jazz and rock music are constructed around a main tonal center, while classical music presents many modulations, so the tonal centers vary with time. In the following table we include some of the obtained values in this work compared with others taken from the case of languages and genetic sequences:

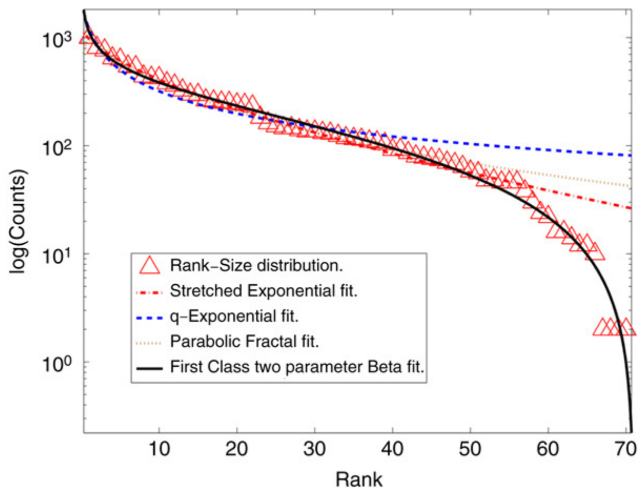

Fig. 4. Semilog plot of the rank-size distribution of Brahms' Symphony No. 4, 1st movement with superimposed fittings. the fittings to be compared are a stretched exponential, a q-exponential, a parabolic fractal distribution, and the first class two-parameter Beta distribution. Note the correct qualitative behavior of the TPBF in the rightmost region of the distribution.

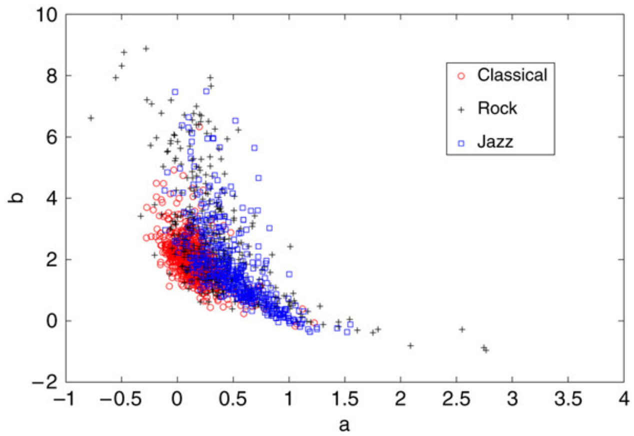

Fig. 5. Fitting parameters a and b plotted as coordinates for classical (circles), jazz (x), rock (cross) and rock without percussion (squares).

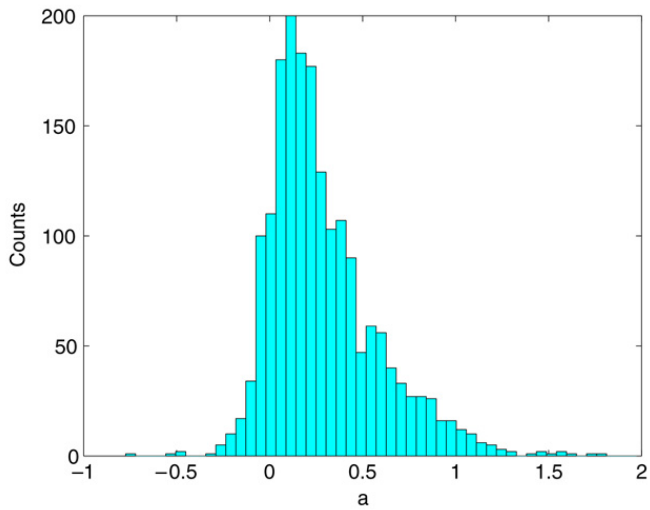

Fig. 6. Distribution of parameter a for the 1800 pieces analyzed.

From Figs. 5, 6 and 7 it is clear that in general $a < b$ for most of the pieces, although some exceptions are observed.

Furthermore, the fact that the beta-like function is observed in music means that behind the statistics of music, there is an underlying multiplicative process, as explained in Ref. [12] for many different systems. However, to gain insight about how this law arises for the particular case of music, let us consider with more detail the note statistics. The "musical code",

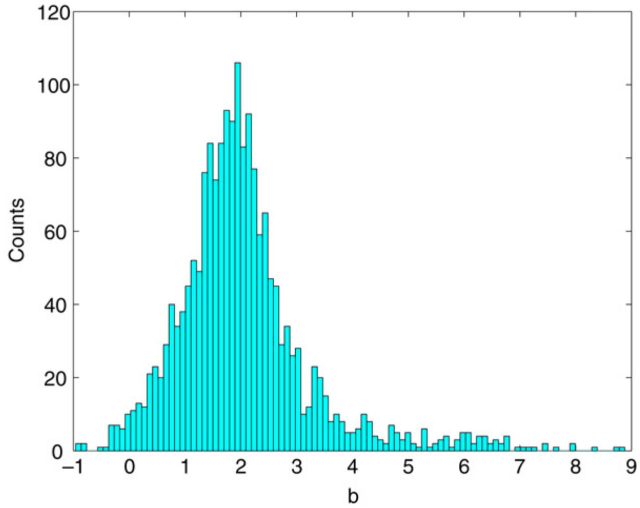

Fig. 7. Distribution of parameter *b* for the 1800 pieces analyzed.

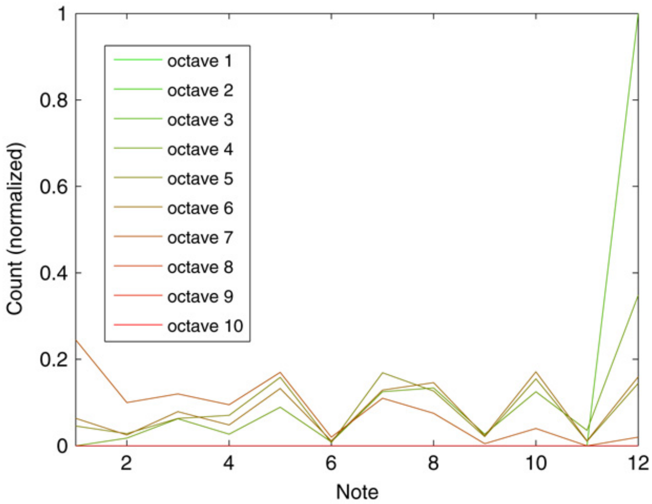

Fig. 8. Histogram per octave of the number of times that a certain note is used. The histogram is normalized to the frequency of each note in a certain octave. The number 1 note is C# and number 12 is C. Notice that F#, A and B are rarely used, as can be expected from harmonic arguments. The piece is Bach's prelude BWV 881 in F minor.

in which all the pieces studied were written in, was originally developed according to a set of conceptions about harmony that were later revised to form the so-called "well tempered scale". This set of notes is arranged in subsets called octaves. Each octave comprises twelve notes, which are given one twelve "names" (C, C#, D, Eb, E,...), each tag assigns the note to a definite role, regardless of the octave it belongs to according to the harmonic scheme of the composition is written in. So the "musical code" if one makes no distinction between two notes that have the same role (i.e. name) but have different sounds owing to different octaves. In such reduced scheme, note 1 corresponds to C#, note 2 to D and so on. Fig. 8 shows the resulting frequency histogram of the notes for Bach's prelude BWV 881 in F minor, divided by octaves. The histogram for each octave is normalized to one. Several features are observed. First one can see that in octave 1 only the note C is used, corresponding to the lowest note available. But the most important observation is that different octaves present only slightly different patterns. For example, notes F#, A and B are almost not used in all octaves, a fact that can be explained in a pure tonal basis, since for example F# is neither contained in the F minor scale nor in other neighbouring tonal centers. We have verified that many other pieces present a similar pattern.

Combining the previous observation with Figs. 1 and 8, it is possible to deduce that the ranking of notes has two different sources. The first is a general envelope given by the octave in which notes are played, since in Fig. 1 it is clear that the most used notes are at the central octaves. We have verified that most of the pieces have this property, although at the moment we can only speculate about this particular distribution (like for example, that the human voice coincides with that part of the spectrum, etc.). The second source of ranking has a harmonic nature, as shown in Fig. 8.

An interesting corroboration of the last assertion, that the total histogram of notes is the product of an envelope and an octave histogram, is the fact that a certain trend is observed for the values of *a* and *b* for major and minor modes, when we compare pieces that have the same instrumentation and musical structure. In Fig. 9 we present the a plot of the type

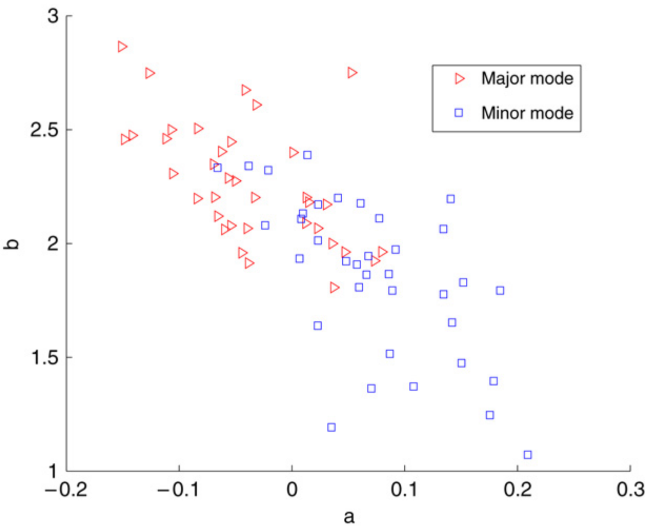

Fig. 9. A comparison of the a and b parameters for Bach's well tempered clavier distinguishing minor and major modes.

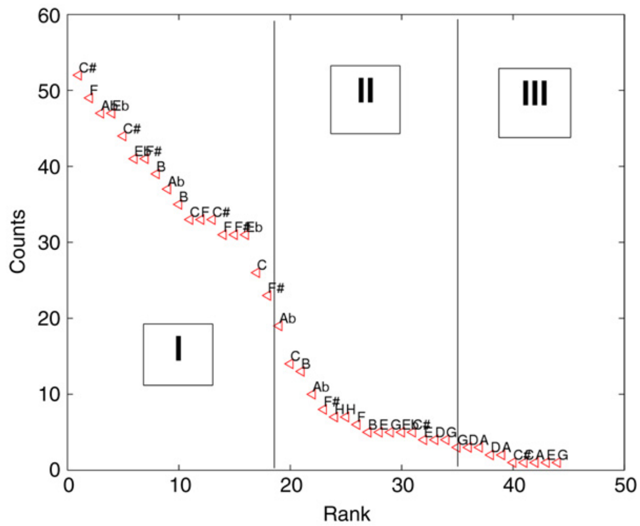

Fig. 10. The regions of the rank-size distribution of a fugue in *C#* Major, Region *I* Contains mostly notes that belong to the Tonal scale, region *III* is characterized by having mostly notes that do not belong to the scale and region *II* is a crossover section.

(a, b) for Bach's well tempered clavier. The triangles are major modes while the squares are minor modes. Note that there is a tendency of minor modes to have higher values of a, which leads to the idea that the values of (a, b) have something to do with the tonal structure of the pieces. And indeed, after some analysis it is noted that the ranked distribution of notes is composed of three sections. The first is made up exclusively of notes that belong to the tonal scale, the third is comprised only of notes that are not in the tonal scale, the middle zone is a mixture of the former two, see Fig. 10. These three sections are a consequence of the degeneration on notes by having both a favorable harmonic role (i.e. are in the tonal scale) and in central octave, just one of these characteristics or none. The subset of notes that are not included in the major scale has much less importance in major modes than the set of notes not used in the minor scale in minor modes, as shown by comparing the twelve-note rank-size distribution of minor and major modes shown in Figs. 11 and 12. Thus, in major modes the middle section of the full rank-size distribution is much smaller than that of the pieces written in minor mode, and the fittings tend in major modes to a smaller (even negative) a parameter to compensate for the bigger b that is needed to fit the narrower gap between the first and third sections.

3. Conclusions

We have shown that a Beta rank law can be used to improve upon the Zipf law, which has been widely used in music, even as a source to test music aesthetics [11]. Such law has its origins in multiplicative processes, like in decision trees [12]. In the particular case of music, it seems that ranking has two contributions, one is due to the position of the notes in the sound spectrum and the second is related with the harmonic nature of a piece. Such effect has a clear impact upon the parameters of the Beta rank law, since a clear distinction is observed for minor and major modes. These results suggest

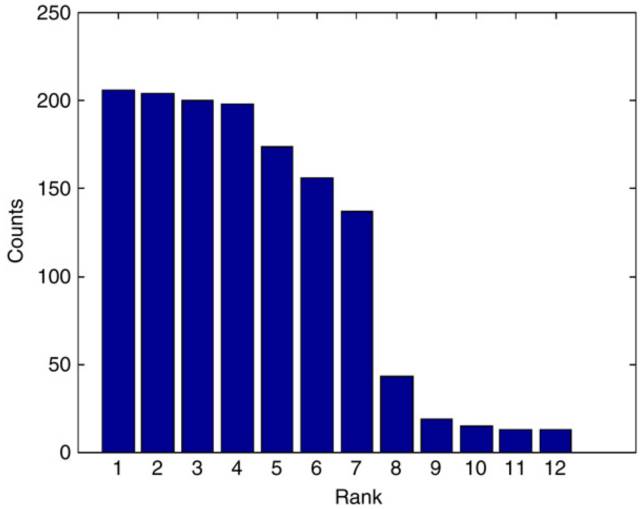

Fig. 11. Typical twelve-note rank-size distribution of a composition written in major mode. The sharp cut after the seventh note is responsible for the small intermediate region in the full rank-size distribution.

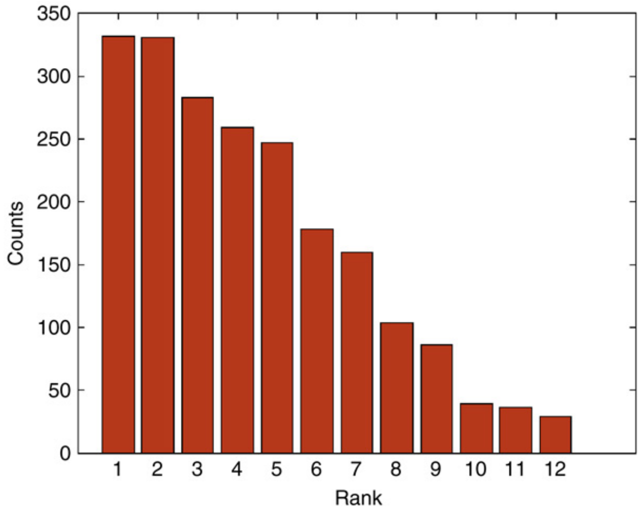

Fig. 12. Typical twelve-note rank-size distribution of a composition written in minor mode. Due to clear harmonic reasons, this distribution is smoother than that of major modes.

the presence of generic brain constraints associated to features of brain neural networks. Perhaps, important aspects of the dynamics of these neural networks could be modeled by the multiplication of number sequences [18]. In absence of external modulations, this behavior would be dominant, restricting the "free inspiration" of the musician. However, if the composers take into account these restrictions, they could follow or violate these restrictions, adding an extra dimension to music composition.

Acknowledgments

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