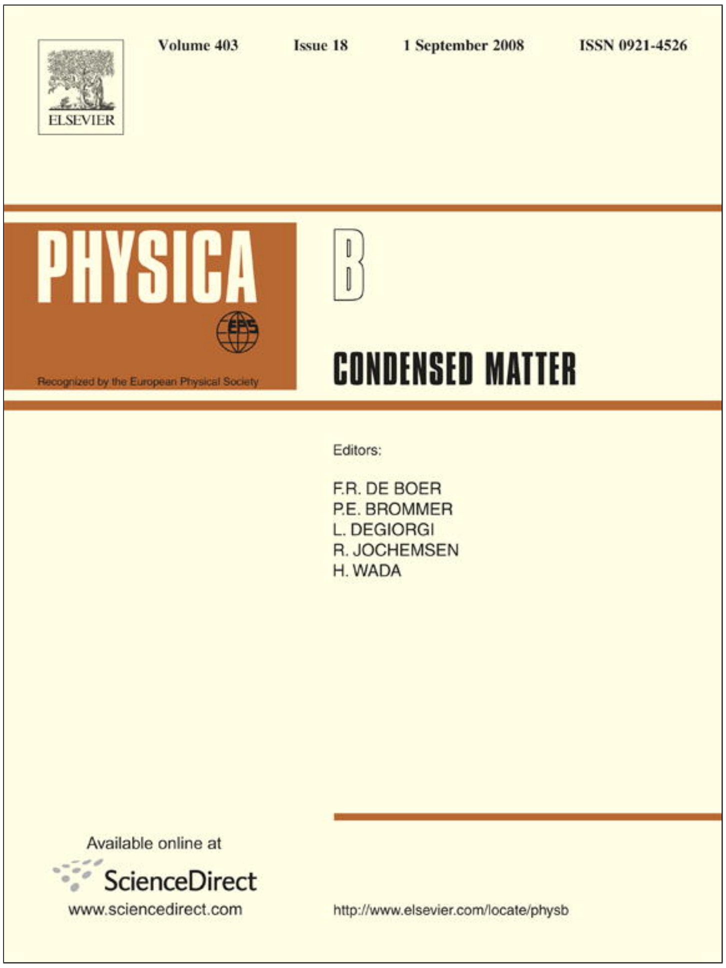
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Reflectance distribution in optimal transmittance cavities: The remains of a higher dimensional space

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ABSTRACT

One of the few examples in which the physical properties of an incommensurable system reflect an underlying higher dimensionality is presented. Specifically, we show that the reflectivity distribution of an incommensurable one-dimensional cavity is given by the density of states of a tight-binding Hamiltonian in a two-dimensional triangular lattice. Such effect is due to an independent phase decoupling of the scattered waves, produced by the incommensurable nature of the system, which mimics a random noise generator. This principle can be applied to design a cavity that avoids resonant reflections for almost any incident wave. An optical analogy, by using three mirrors with incommensurable distances between them, is also presented. Such array produces a countable infinite fractal set of reflections, a phenomena which is opposite to the effect of optical invisibility.

1. Introduction

Although it is well known that quasiperiodic structures are obtained as projections from higher dimensional spaces [1], it has not been possible to obtain their physical properties as projections or enlargements of the dimensionality [2]. Such is the case of quasicrystals (crystals with forbidden symmetry) in which the structure is described using a higher dimensional periodic crystal [1]. It was thought that this simplification in the description of the geometry of quasicrystals could have been useful to deduce physical properties, like the electronic conduction, since Bloch's theorem can be applied in the higher dimensional space. This, however, has not been possible and theories of physical properties make no reference to the underlying periodic lattice [2–4].

In this work we present a simple problem that yields a physical property that displays the remaining traces of a higher dimensional space due to the existence of two incommensurate length scales. The present work is thus similar to the recent trend in the research of quasicrystalline photonic bandgap arrays, which is generating a lot of interest in the optics community but which has grown out of basic work in quasicrystals [5].

This proposed problem is a one-dimensional cavity with three barriers which can be tuned to avoid resonant reflections for almost any incident wave that fulfills a Helmholtz equation. We show how this apparently simple problem, which at first sight seems to be a sort of undergraduate exercise, leads to a reflection and transmission probability density given by the density of states (DOS) of a tight-binding Hamiltonian defined over a two-dimensional triangular lattice. As we shall prove, this effect is due to a phase decoupling of the scattered waves produced by the incommensurable length scales of the system. By phase decoupling we mean that, in principle, the transmittance of the cavity is given by the coherent interference of scattered waves whose wave vectors are multiples of a fundamental wave-length. However, when the cavity is incommensurate, the interference between the scattered waves occurs in an incoherent way. We prove this assertion by showing that the phase differences between the scattered waves are almost random, having in mind the fact that quasiperiodic functions can be used in an efficient way as a random noise generator. Since we can expect cavities in acoustics, optics, wave mechanics, electronics, etc., the presented results are quite general. It is clear that one can study much more sophisticated systems, but the proposed example provides the

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most simple example of phase decoupling without loosing the essential feature: the almost random nature of a system made with incommensurate length scales. Interestingly, an elementary example slightly different to the one presented here (a double barrier potential with adjusted parameters) allowed the exhibition of a degeneracy of resonances and the occurrence of complex double poles in the scattering matrix of the problem [6].

The idea of phase decoupling due to incommensurability provides also a way to produce useful effects, like avoiding reflection resonances at interphases or in the generation of infinite images of an object. As an example, we propose an optical device that illustrates the effect: three mirrors separated at incommensurable distances, producing a fractal infinite set of reflections. This multiplicity can be used in some applications as, for example, to produce many radial images of an object, or in some kind of fun-house mirror effects (which is a reminiscent of a trick used in the movies). In that sense, the present phenomena is the opposite of optical invisibility [7,8], although one can also hide an object between its multiple images. Such trick was presented in a masterful way by the director Orson Welles in the famous labyrinth of mirror's fight scene from the film "The lady from Shanghai".

Another possible application of the methodology presented here can be the use of an alternative understanding of the conductance statistics in cavities, which is a relevant topic in mesoscopic physics [9] and in nanotechnology [10]. The model presented, although being simple, shows that the behavior of the transmittance can be obtained analytically in terms of an underlying higher dimensionality. It is worthwhile mentioning that there is no chaos in one-dimensional, so the model presented here does not allow to address such problem; however, it points to a possible new interpretation based in an extended dimensionality of the problem, alternative to the successful approach of random matrix theory [11]. Notice that there is a possible link between these two approaches, since for incommensurate systems, one can use the transfer matrix formalism [12–15]. The main difference is that in the former case, the matrices are not random. However, the present article suggests that they can mimic a random sequence of matrices.

We have also to mention that although a lot of work has been done in studying the reflectivity of quasiperiodic sequences [16–18], like in the Fibonacci chain [19], our results differ in the sense that we study a simple cavity and not an infinite sequence of scatters. This simplification produces an analytical formula for the reflectivity, something which was not possible in the previous approaches. Also, the present approach opens the door for the use of a higher dimensional method to study the reflectance and transmittance statistical properties of complex sequences of scatters.

The outline of this paper is the following. In Section 2 we provide the required geometry of the cavity and the equations. Section 3 is devoted to the optimization of the resonator which eventually leads to the discussion of the phase decoupling phenomena, the mirror effect is discussed in Section 4 and, finally, concluding remarks are presented in Section 5.

2. A generic simple cavity model

The basic idea of a resonator, active or passive, is to control absorption or losses in such a way that the reflection and transmission could be related to the geometry of the confining cavity. The wave length and the phase are related to the dimension of the cavity and are used to determine the transmission and the reflection of the wave. For instance, such idea can be used to build an etalon, which is an optical instrument that measures wavelengths, i.e., is an spectroscopic device that has two flat parallel reflecting surfaces used to measure wavelengths through interference. In optics, two or more etalons have been used for some time as an experimental rule for white-light rejection [20,21], and to assure no coincidence of etalon pass-bands over a substantial frequency range. The ratio of etalon optical gaps must be given by a ratio of two mutually prime integers [20,21]. Of equally technological importance is the optimization of periodically poled electrical domains or any other phenomena that can be understood as a succession of cavities or determined by the interaction between cavities [22]. Here the word poled has the meaning of the result of applying a strong electric field over a material. A strong electric field can order or align a molecule or electric domain in a material accordingly to such field.

Following these kind of ideas, suppose that we want to build a simple model for a cavity or interphase in order to make it smooth in the sense that almost all frequencies are reflected in the same way. First we construct a simple resonator by considering the generic Helmholtz equation that describes wave propagation in acoustics, optics, wave mechanics, fluids in the shallow water regime, etc. [23]. The corresponding stationary equation, say for the quantum case, is of the type,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{d^2x}+V(x)\psi(x)=E\psi(x), \quad (1)$$

where $\psi(x)$ is the wave function at site x, $V(x)$ the potential and E the eigenvalue. When $E>V(x)$, the solutions are propagating waves of the type $\exp(ikx)$. Let us suppose that a wave is traveling from left to right as shown in Fig. 1a, approaching certain obstacles, like for example membranes or a repulsive potential. A simple model is to consider the potential as a Dirac delta centered at site x_0, where $V(x)=\alpha\delta(x-x_0)$. A delta potential centered at x_0 scatters the incident wave-function by an amount which is directly proportional to the reflected wave. The scattered wave is given by

$$\psi_R(x)=\mathscr{S}(x_0)\psi(x),$$

where the scattering factor $\mathscr{S}(x_0)$ is given by

$$\mathscr{S}(x_0)=\frac{i\beta}{1-i\beta}\exp(2ikx_0), \quad (2)$$

where $\beta=m\alpha^2/E\hbar^2$ and k is the wave vector of the incident wave. To build the cavity, three delta functions centered at sites x_1, x_2 and x_3 are used, i.e., the potential is given by

$$V(x)=\alpha\delta(x-x_1)+\alpha\delta(x-x_2)+\alpha\delta(x-x_3). \quad (3)$$

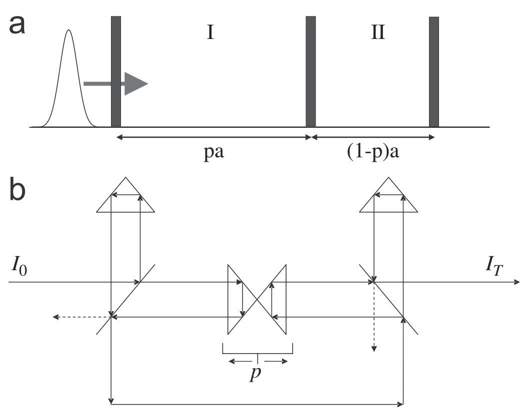

Fig. 1. Geometry of the cavity. The position of the second scatter is regulated by the parameter p: (a) electronic image and (b) Tandem Michelson interferometer.

In the limit $E \gg m\alpha^2/2\hbar^2$, we have $\beta \ll 1$ and small reflection amplitudes. Thus, the total scattering can be approximated as the sum of the scattering of each delta potential separately. Without loss of generality, if the size of the cavity is a, then we set $x_1 = 0$, $x_3 = a$ and $x_2 = pa$, where p is a number between 0 and 1. The number p can be used as a control parameter of the position of the second scatter inside the cavity. In such case, the reflection coefficient of the cavity, $R(k,p)$, as a function of the incident wave vector k and p, is the norm of the sum of scattering given by Eq. (2), with the proper normalization factor:

$$R(k,p) \equiv \frac{\beta^2(3 + 2\cos(2ka) + 2\cos(2pka) + 2\cos(2(1-p)ka))}{4(1+\beta^2)}. \quad (4)$$

A simple way to reinforce the reflection for a given k, is to diffract the wave by setting $p = \frac{1}{2}$ and $a = \lambda$, where λ is the wavelength. Fig. 2a shows a contour plot of $R(k,p)$, which clearly shows that the minimum occurs for $p = \frac{1}{2}$. The same effect is observed for $k = 2\pi n/a$, where n is an integer. A cavity that scatters in this wavelength produces zero reflectance for other wavelengths, as for example a wave with $k' = \pi/a$ and all the odd harmonics of this wave will pass thorough the cavity without being blocked. The relevant point here is to know which p produces the optimal result to avoid strong reflections.

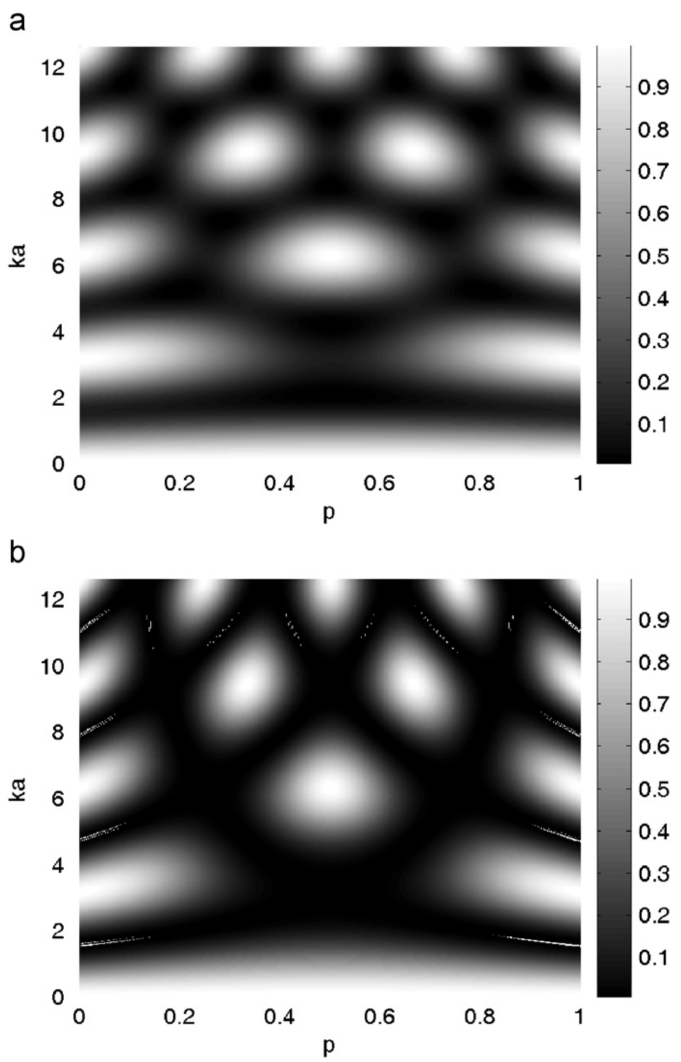

Fig. 2. Density plot of the reflectance, as a function of the geometrical control parameter p and wave-number ka, for: (a) cavity with Dirac deltas and (b) Tandem Michelson interferometer. The color code for the reflectance appears at the right.

Before giving an answer to the question, we shall discuss the optical version of the resonator since its realization may produce the same effects as the one proposed, but with a more accessible experimental confirmation. A basic instrument that converts a optical path difference in fringe patterns is the optical interferometer. One amplitude division interferometer that uses the interference between only two beams is the Michelson interferometer. To produce transmission, we use it in tandem, with mutually related phase difference, as can be seen in the Fig. 1b. Assuming that the beam splitter is 50% and with no losses, we have that

$$I_T(k,p) = I_0 \cos^2(kpa)\cos^2((1-p)ka). \quad (5)$$

The plot of $I_T(k,p)/I_0$ is shown in Fig. 2b.

3. The reflectance distribution of an optimal cavity

Now let us go back to the originally posed question: Which p produces the optimal result to avoid strong reflections at almost all frequencies? As we shall see, answering this question will eventually lead us to an extension of the dimensionality of the problem. Furthermore, the involved design can be very useful in optics and in any system in which impedance match from incoming waves is desired, like in acoustics, fluids, wave guides, etc. Let us first give an intuitive response to the proposed question. The system can be viewed as two square wells that interact through a perturbation and maximal reflection occurs when the diffraction condition holds in each of the cavities, and standing waves are produced inside each region. The conditions of having standing waves in region I and II of the device are, $k_1 = 2\pi n/pa$ and $k_2 = 2\pi l/(1-p)a$, where n and l are integers. The ratio k_1/k_2 determines the mismatch:

$$\frac{k_1}{k_2} = \frac{l(1-p)}{np}. \quad (6)$$

The best compromise is to take p as an irrational number to avoid such resonance. A different way of understanding this phenomenon is to observe that the eigenfunctions with a given wavelength in region I will never satisfy the boundary conditions in region II. Actually, in this case the spectrum is built upon two series of levels, $E_1 \approx -2\pi^2\hbar n^2/mp^2a^2$ and $E_2 \approx -2\pi^2\hbar n^2/m(1-p)^2a^2$, which have incommensurate spacing between them. If p is incommensurate, then it is clear from Eqs. (4) and (5), that the reflectance is a quasiperiodic function of k. In Fig. 2, this corresponds to a cut at an irrational p. Since a good mismatch between the two regions is needed, it seems that the most irrational number will perform better. The inverse golden section $1/\tau = 2/(\sqrt{5}+1) = 0.618034\ldots$ is the best choice since its corresponding continued fractions representation converges at the slowest pace among all irrationals. The corresponding rational approximants are given by the ratio of two successive Fibonacci numbers. An inspection of Fig. 2 suggests that this number can work well, since values close to this number, such as $p = 0.62$, produce few maximal or minimal reflections.

Now let us quantify how effective is the reflector. This can be done by calculating the distribution of reflectance $P(R)$ for a given p. To do so, the statistics of the reflectance produced by Eqs. (4) and (5) can be calculated using a big cutoff k_c such that $k_c \gg 2\pi/a$. Fig. 3a shows examples of this function calculated numerically for different values of p (notice that the problem is symmetric with respect to $p = \frac{1}{2}$) for $k_c = 20\,000(2\pi/a)$. The most appealing result is that for $p = 1/\tau$, the $P(R)$ is similar to the DOS of a tight-binding Hamiltonian defined in a triangular lattice [24,25]. For $p = 1$, the obtained curve is just the DOS of a one-dimensional tight-binding Hamiltonian. Such result is expected since for $p = 1$, Eq. (4) turns

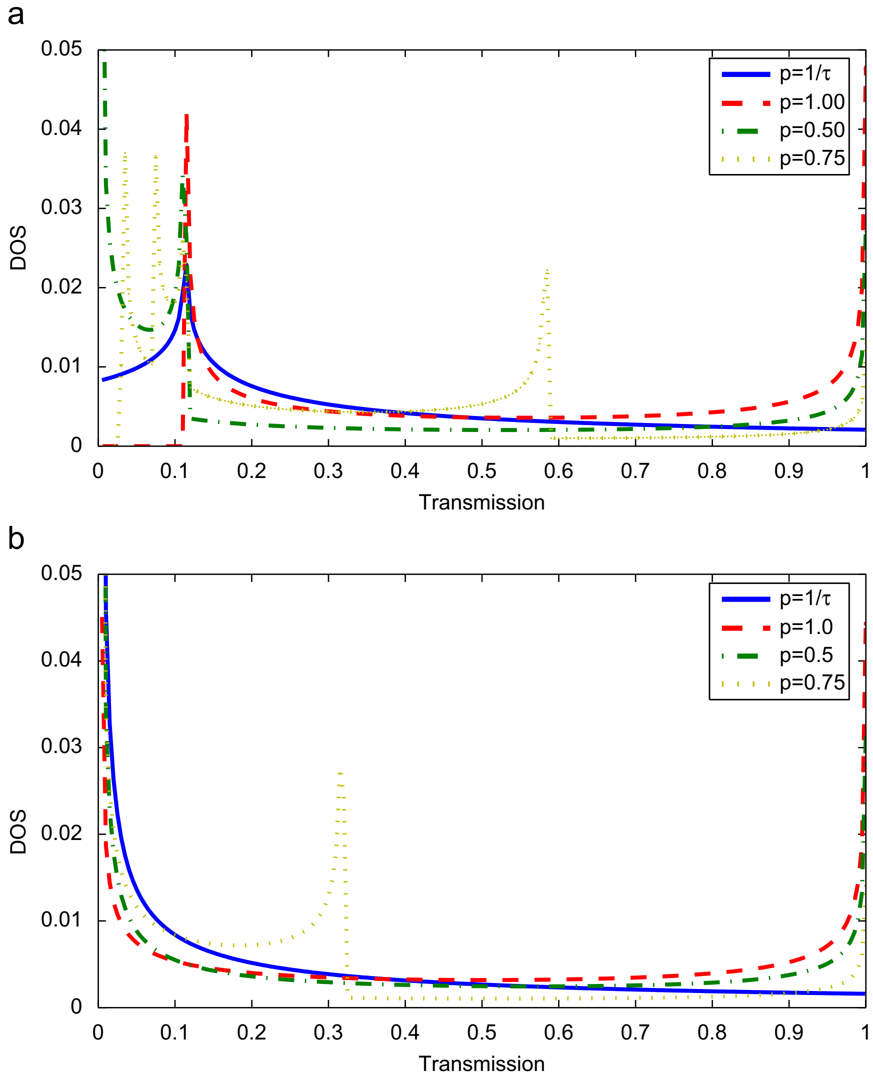

Fig. 3. Distribution function of reflectances for different configurations of the cavity formed by Dirac deltas and (b) the same but for the Tandem Michelson interferometer. Observe that the incommensurate ratio shows only one resonant peak.

out to be the dispersion relation of a one-dimensional chain. A similar idea works for $p = \frac{1}{2}$.

3.1. *The remains of the triangular lattice*

Here we shall show that the relationship between Eq. (4) and the DOS of the triangular lattice is not a coincidence, since the effective dimensionality is increased due to the quasistatistical independence of the phases in $\cos ka$ with respect to $\cos pka$ when p is an irrational.

Let us first observe that any real number z can always be written as $z = \lfloor z \rfloor + \{z\}$, where $\lfloor z \rfloor$ is the integer part of a number and $\{z\}$ the fractional part. Thus,

$$\cos 2z = \cos\left(2\pi\left(\left\lfloor \frac{z}{\pi} \right\rfloor + \left\{\frac{z}{\pi}\right\}\right)\right) = \cos\left(2\pi\left\{\frac{z}{\pi}\right\}\right). \tag{7}$$

Applying this identity, it follows that $\cos(2ka) = \cos(2\pi\{ka/\pi\})$ and $\cos(2pka) = \cos(2\pi\{pka/\pi\})$. Since the function $\{z\}$ is periodic with period 1, when p is irrational, $\{pka/\pi\}$ turns out to be incommensurate with respect to $\{ka/\pi\}$. Thus, $|\{ka/\pi\} - \{pka/\pi\}|$ is a quasiperiodic function that fills densely the interval $[0, 1)$. In fact, this is a well known procedure to get a pseudo-random number generator [26]. As a result, we can treat $\{pka/\pi\}$ and $\{ka/\pi\}$ as if they were almost independent random variables when k is changed. Let us call these variables $\chi_1(k)$ and $\chi_2(k)$, respectively; both of them take values between 0 and 1, with an uniform distribution. Having in mind the previous discussion, we have that

$$\begin{aligned}\cos(2(1-p)ka) &= \cos(2ka)\cos(2kap) + \sin(2ka)\sin(2kap)\\ &= \cos(2\pi(\{pka/\pi\} - \{ka/\pi\}))\\ &= \cos(2\pi(\chi_1(k) - \chi_2(k))).\end{aligned} \tag{8}$$

The final step is to combine Eqs. (7) and (8) with the equation for reflectance (4), which leads to

$$R(k,p) = \frac{\beta^2[3 + 2\cos(2\pi\chi_1(k)) + 2\cos(2\pi\chi_2(k)) + 2\cos(2\pi(\chi_1(k) + \chi_2(k)))]}{4(1+\beta^2)} \tag{9}$$

The phases of the cosine functions, containing with $\chi_1(k)$ and $\chi_2(k)$, behave as almost random independent variables, a phenomena that we call phase decoupling. Thus, we define two wave vectors $k_x \equiv 2\pi\chi_1(k)$ and $k_y \equiv 2\pi\chi_2(k)$ which are independent variables. Using the previous definition, $R(k,p)$ can be renamed as an effective two-dimensional reflectivity $R_{\mathrm{eff}}(k_x, k_y) \equiv R(k,p)$. Using Eq. (9), $R_{\mathrm{eff}}(k_x, k_y)$ is written as

$$R_{\mathrm{eff}}(k_x, k_y) = \frac{\beta^2(3 + 2\cos(k_x) + 2\cos(k_y) + 2\cos(k_x + k_y))}{4(1+\beta^2)}. \tag{10}$$

It turns out that $R_{\rm eff}(k_x,k_y,p)$ is the dispersion relation of a two-dimensional triangular lattice Hamiltonian with first neighbor interaction. Indeed, consider a simple s-band tight-binding Hamiltonian, in which the stationary Schröedinger equation is [27]

$$E\psi_l = V(l)\psi_l + \sum_{\langle j,l\rangle} t\psi_j, \tag{11}$$

where ψ_l is the wave-function at a given atom l, t is the resonance integral between atomic sites l and j, and $V(l)$ the on-site potential. The symbol $\langle j,l\rangle$ indicates that the sum must be carried over nearest neighbor atoms; all other interactions are considered negligible [27]. The geometry in which the Hamiltonian is defined can be chosen as a triangular lattice, as suggested by Eq. (10). Since the triangular lattice is periodic, we can use the Bloch's theorem to propose a periodic solution of the type,

$$\psi_j = \exp(\mathbf{k}\cdot\mathbf{r}_j), \tag{12}$$

where $\mathbf{r}_j$ is the position of the atom j in the triangular lattice and $\mathbf{k}$ is a wave vector in two dimensions with components (k_x,k_y). The positions of the atoms are given by linear combinations $\mathbf{r}_j = h_j\mathbf{e}_1 + l_j\mathbf{e}_2$ of the triangular lattice basis vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (1,1)$, where h_j and l_j are any integers and the lattice parameter is taken as unity. Notice that apparently, the basis vectors are different from a perfect triangular lattice; however, it is easy to prove that the Hamiltonian given by Eq. (11) takes into account only the topological connectivity of the lattice, so that ultimately the particular choice of the basis turns out to be irrelevant provided that the topological connectivity is maintained. By inserting the proposed solution into Eq. (11), and taking into account the appropriate phase for each of the six first neighbors of the triangular lattice, we obtain the dispersion relation of the two-dimensional triangular lattice,

$$\begin{aligned} E = V + t[&\exp(k_x) + \exp(-k_x) + \exp(k_y) + \exp(-k_y) \\ &+ \exp(k_x+k_y) + \exp(-k_x-k_y)]. \end{aligned} \tag{13}$$

By setting $t = \beta^2/2(1+\beta^2)$ and the self-energy as $V = 3\beta^2/4(1+\beta^2)$, we recover Eq. (10). Thus, we have shown that the reflectance of the two-dimensional lattice is equal to the energy of the two-dimensional tight-binding Hamiltonian, i.e., $E = R_{\rm eff}(k_x,k_y)$.

Notice also that since the DOS in the electronic case, $\rho(E)$, is the number of wave vectors $\mathbf{k}$ having an energy E, and $P(R)$ is the number of wave vectors $\mathbf{k}$ that produce a reflectivity $R_{\rm eff}(k_x,k_y)$, it follows that $\rho(E) = P(R)$.

There are many well-known procedures to generate the DOS starting from a dispersion relation, for example, one can use the integral of the norm of the inverse group velocity taken over isoenergetic surfaces, $S(E)$, in the first Brillouin zone [27]:

$$\begin{aligned} \rho(E) = P(R) &= \frac{1}{2\pi^2}\int_{S(E)} \frac{1}{\|\nabla_{\mathbf{k}}E\|}\,\mathrm{d}k_x\,\mathrm{d}k_y = \frac{(1+\beta^2)}{\pi^2\beta^2} \\ &\times \int_{S(E)} \frac{\mathrm{d}k_x\,\mathrm{d}k_y}{\sqrt{[\sin k_x + \sin(k_x+k_y)]^2 + [\sin k_x + \sin(k_x+k_y)]^2}}. \end{aligned} \tag{14}$$

An equivalent procedure consists into calculate the Green's function $G(l,j,E)$ at sites l and j of the Hamiltonian,

$$G(l,j,E) = \frac{(1+\beta^2)}{\pi^2\beta^2}\int_0^{\pi}\int_0^{\pi} \frac{\cos(k_x l)\cos(k_x j)\,\mathrm{d}k_x\,\mathrm{d}k_y}{E - (\cos(2k_x) + 2\cos(k_x)\cos(k_y))}, \tag{15}$$

and then use the identity [28],

$$\rho(E) = -\frac{1}{\pi}\mathrm{Im}\sum_l G(l,l,E). \tag{16}$$

Eqs. (15) and (16) were already calculated some years ago by Horiguchi [24], who obtained

$$\rho(E) = \frac{(1+\beta^2)}{\pi^2\beta^2}A\mathbf{K}(\sqrt{1-u^2}),$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind. A is defined as:

$$A = \frac{8}{(\sqrt{2E+3}-1)^{3/2}(\sqrt{2E+3}+3)^{1/2}}$$

and $u = A(2E+3)^{1/4}/2$.

A corroboration of all the above results is that the graphics of $P(R)$ shown in Fig. 3a is exactly the same that the DOS of the triangular lattice obtained in the work by Sakaji et al., [25]. Furthermore, some features of the $P(R)$ are easy to understand in terms of the proposed analogy. For example, the lone peak observed in $P(R)$ for $p = 1/\tau$ has a simple explanation. In the electronic case, the maxima of the reflectivity is due to a Van Hove singularity which occurs when the group velocity $\nabla_{\mathbf{k}}E$ is zero [27]. The Van Hove singularity arises because the eigenfunctions turn out to be stationary due that they have the same periodicity of the lattice, i.e., the corresponding isoenergetic touches the limit of the Brillouin zone. The compression to the left of $P(R)$ in Fig. 3a is due to the odd rings in the Hamiltonian that produce frustration for antibonding electronic states.

It is interesting to observe how the two-dimensional approach allows us to construct a two-dimensional reciprocal space for the problem. All these conclusions arise from the fact that phases are decoupled. We must point out that all of the previous steps are reversible, i.e., given a two-dimensional Hamiltonian, one can find an incommensurate one-dimensional system that has a reflectance statistics similar to the dispersion relationship. It is worthwhile to remark that other cavities lead to different lattice topologies. Finally, Fig. 3b presents the DOS for a tandem Michelson; the result is similar to Fig. 3a, except that the most frequent value is zero.

4. The incommensurate array mirror effect

The absence of resonances for the irrational case and the extension of the dimensionality can be explained in terms of simple optical mirrors, in which an infinite number of images are produced when an object is placed between the mirrors. Each image can be associated with a point in a two-dimensional space, and since each image is produced in different places, there is no possibility of enhancing a given reflection. To clarify this idea, let us consider three plane mirrors facing each other in a one-dimensional array with an arrangement similar with those of the cavity made from delta functions (Fig. 1a). The mirror at the middle is semi-transparent, so a fraction of the light is transmitted and the other reflected. The reflections of an object inside such device are calculated by obtaining all the mirrors that are produced by successive reflections. The n-th mirror at position x_n, can be reflected in the m-mirror at position x_m, producing a new mirror due to the following symmetry operation: $\mathscr{M}_{x_m}(x_n) = 2x_m - x_n$. The initial position of a mirror (say x_n) can be written as a linear combination generated by pa and a:

$$x_n = h_n a + l_n pa, \tag{17}$$

where h_n and l_n are integers. These numbers define a point in a two-dimensional lattice with coordinates given by (h_n,l_n). The first three initial planes have integer coordinates $(0,0)$,$(1,0)$ and $(0,1)$. When two planes at positions of the form $h_n a + l_n pa$ are reflected

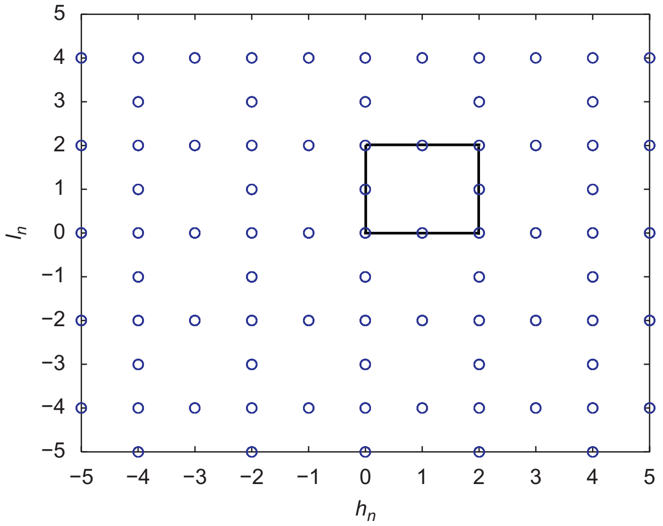

Fig. 4. Two dimensional representation of the distribution of mirror planes for the cavity discussed in the text. Each circle denotes the coefficients of a possible reflection, at the position $x = h_n a + l_n pa$. The unit cell is a square with a missing central point.

by each other, we get that,

$$\mathscr{M}_{x_m}(x_n) = (2h_m - h_n)pa - (2l_m - l_n)a. \tag{18}$$

This shows that the new mirror at $\mathscr{M}_{x_m}(x_n)$ is also a linear combination of a and pa. Formally, all linear integer combinations of a and pa densely fill the real plane, so a dense infinite number of images are expected. Notice, however, that if we begin to apply $\mathscr{M}_{x_m}(x_n)$ to the three initial mirrors, then it is observed that not all the linear combinations are obtained. For example, the point $(1,1)$ is never reached. We can represent all mirrors planes in a two-dimensional square lattice with coordinates (h_n, l_n), as shown in Fig. 4. According to the previous discussion, however, the unit cell is a square with a missing point at the center and due to this, only $\frac{3}{4}$ of the square lattice vertices are integer coordinates of the reflection planes. The consequence is that the resultant set of reflections of a given object, turns out to be an infinite set, with a fractal nature.

5. Concluding remarks

In summary, we study a incommensurate cavity that offers an appealing example in which the physical properties are determined by a higher dimensional Hamiltonian. Such effect is due to a phase decoupling produced by the pseudo-random behavior of quasiperiodic functions. The proposed method provides a mechanism to produce an infinite fractal set of images of an object and gives also clues to design useful devices to avoid reflection resonances, and gives a simple approach to the transmission and reflection properties of cavities.

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