Analytic expressions for the vertex coordinates of quasiperiodic lattices

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Abstract. Using the generalized dual method, closed analytical expressions for the coordinates of quasiperiodic lattices, derived from periodic or quasiperiodic grids, are given. The obtained formulae constitute a useful and practical tool to generate and perform calculations in quasiperiodic structures.

1. Introduction

The most general and most frequently applied methods to generate quasiperiodic tilings are the cut and projection method [1] and the generalized dual method (GDM) [2, 3]. The former is based on the idea, introduced by H. Bohr [4], that a quasiperiodic function in $\mathbb{R}^n$ can be obtained by adequately restricting a periodic function with N variables, $N > n$. The cut and projection method then obtains a quasiperiodic lattice in $\mathbb{R}^n$ by projecting points of a periodic lattice in $\mathbb{R}^N$. The GDM is a generalization of the method proposed by de Bruijn [5] to generate the vertex coordinates of the Penrose tiling by means of a pentagonal multigrid. In the GDM, the multigrid is generated as a family of planes normal to a set of vectors with definite orientational symmetry, called the star-vector. Vertex coordinates of the quasilattice are generated through a dualization process, which associate vertices with regions between the planes of the multigrid. Multigrids with arbitrary orientational symmetry and spacing (periodic or quasiperiodic) can be considered. All the structures generated by the cut and projection method can also be obtained by the GDM, but the converse is not true [6]. In that sense, the GDM is more general.

Both methods can be computationally implemented to generate quasiperiodic structures, with due consideration of some practical difficulties. In the cut and projection method the main problem is to determine if a given point of an N-dimensional lattice falls inside a well defined region in $\mathbb{R}^N$, the strip. General solutions based on linear programming [7] and inequalities [8] have been proposed. In the GDM, it turns out that a given multigrid can contain very small regions which makes the dualization procedure dependent of the machine precision.

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Based on the GDM, we obtain in this work analytical expressions of the dualization process that lead us to provide formulae for the vertex coordinates of a quasiperiodic structure. Though the computational problems with the standard approaches described above can be overcome, and practical computer programs can be easily obtained [9], the availability of analytical expressions for the vertex coordinates of a quasilattice can be of great value. It offers not only a practical tool to easily generate arbitrary quasiperiodic structure but may be useful to perform calculations in quasiperiodic structures in a systematic analytical way. A restricted version of the present approach was used, for instance, to obtain average structures associated with a quasilattice, and to calculate their diffraction properties [10].

2. A brief survey of the GDM

Within the GDM, a quasiperiodic structure is obtained as follows [2, 3]. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N\}$ in $\mathbb{R}^n$ be a star of vectors, which determines the orientational symmetry of the quasilattice (for real quasicrystals, $n = 2, 3$). An N-grid G_N is defined as the union of an infinite set of parallel planes in $\mathbb{R}^3$ (lines in $\mathbb{R}^2$) orthogonal to the vectors $\boldsymbol{e}_i$. In general, the separation between planes can be given by a periodic or quasiperiodic sequence and the array of planes can be translated relative to the origin. The more general expression of an N-grid is

$$G_N = \{\boldsymbol{x} \in \mathbb{R}^3 \boldsymbol{x} \cdot \boldsymbol{e}_j = x_{n_j} = n_j + \alpha_j + \chi_{n_j}\,; \quad j = 1, 2, \ldots, N, n_j \in Z\}\,, \tag{1}$$

where $\alpha_j \in \mathbb{R}$ are shifts of the grid with respect to zero and χ_{n_j} defines the spacing of grid planes along the direction labelled by j. Periodically spaced grids are obtained with $\chi_{n_j} = 0$, for all j. More general structures are obtained by considering a quasiperiodic spacing of grid lines, controlled by:

$$\chi_{n_j} = \frac{1}{\varrho_j} \lfloor n_j \sigma_j + \beta_j \rfloor\,,$$

where $\lfloor\ \rfloor$ is the greatest integer function (or floor function). The position of the n-th line along the direction j is then given by

$$x_{n_j} = n_j + \alpha_j + \frac{1}{\varrho_j} \lfloor n_j \sigma_j + \beta_j \rfloor\,, \tag{2}$$

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where σ_i is an irrational number which controls the sequence and frequency of the two possible intervals between x_n and x_{n+1}. α_i and β_i are shifts of the grid with respect to zero, and ϱ_s determine the ratio of the two different lengths that appears in the sequence.

The grid G_N divides the space into open regions limited by planes (lines). Each point in these spaces can be indexed by a set of N integers corresponding to its ordinal position in the grid (given by n_j) for each $\boldsymbol{e}_j$, that is, if a point in an open region lies between the planes (lines) k_j and $k_j + 1$, for the direction $\boldsymbol{e}_j$, we assign the number k_j to this region. A similar construction applies to each star vector, to obtain the N coordinates $(k_1, k_2, \ldots, k_N)$.

The quasilattice is obtained by using the dual transformation, that maps each open region of the N-grid into a point

$$\boldsymbol{t} = \sum_{j=1}^{N} k_j \boldsymbol{e}_j \,, \tag{3}$$

which lies in $\mathbb{R}^3$ ($\mathbb{R}^2$). The point $\boldsymbol{t}$ is a vertex of a quasiperiodic tiling of rhombic unit cells with orientational symmetry corresponding to the star vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N\}$.

In the next sections, we discuss the dualization process in detail in order to obtain analytic expressions for the vertex coordinates of any quasiperiodic lattice obtained with the above method. We first consider the case of multigrids with periodic spacings and present the most general case in Section 4.

3. Periodically spaced grids

According to Eq. (1), the planes of a multigrid with periodic spacings satisfy:

$$\boldsymbol{x} \cdot \boldsymbol{e}_j = x_{n_j} = n_j + \alpha_j \,,$$

where $\boldsymbol{x}$ is a vector, n_j is an integer and α_j are shifts of the grid with respect to zero.

The vertices $\boldsymbol{t}$ of a quasilattice in $\mathbb{R}^3$ are defined by the intersection of three planes, which produces the eight vertices of a cell when the dual transformation is used. For each combination of star vectors, these intersections are the solutions of the following equations,

$$\boldsymbol{x} \cdot \boldsymbol{e}_s = x_{n_s} \,, \qquad \boldsymbol{x} \cdot \boldsymbol{e}_j = x_{n_j} \,, \qquad \boldsymbol{x} \cdot \boldsymbol{e}_k = x_{n_k} . \tag{4}$$

The solution of this system, $\boldsymbol{x}$, is obtained by using Kramer's rule and is given by a linear combination of three vectors. For a given s, j and k, the solution can be expressed as:

$$\boldsymbol{x} = x_{n_s} \boldsymbol{u}_{ksj} + x_{n_j} \boldsymbol{u}_{sjk} + x_{n_k} \boldsymbol{u}_{jks} \,, \tag{5}$$

where

$$\boldsymbol{u}_{sjk} = \frac{\boldsymbol{e}_j \times \boldsymbol{e}_k}{(\boldsymbol{e}_s \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k))} = \frac{\boldsymbol{e}_j \times \boldsymbol{e}_k}{V_{sjk}} \,,$$

and $V_{sjk} = \boldsymbol{e}_s \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k)$ is the volume of the rhombohedron spanned by the edges $\boldsymbol{e}_s, \boldsymbol{e}_j$ and $\boldsymbol{e}_k$.

For two-dimensional grids we have a similar expression,

$$\boldsymbol{x} = x_{n_j} \boldsymbol{u}_{jk} + x_{n_k} \boldsymbol{u}_{kj} \,, \tag{6}$$

where,

$$\boldsymbol{u}_{jk} = \frac{\boldsymbol{e}_j^{\perp}}{a_{jk}} \,,$$

for a given j and k. $\boldsymbol{e}_j^{\perp}$ is a vector perpendicular to $\boldsymbol{e}_j$, and a_{jk} is the area of the rhombus generated by $\boldsymbol{e}_j$ and $\boldsymbol{e}_k$.

Once the intersection points of the grids are found, the next step is to calculate the dual transformation (3). Using for simplicity the case of a multigrid in $\mathbb{R}^2$, we shall show that a simple expression for the ordinal coordinates k_j can be found. Let us consider the family of lines generated by the vectors $\boldsymbol{e}_j$ and $\boldsymbol{e}_k$. Leaving aside for the moment the shift terms α_j, a close analysis shows that around each intersection, there are four regions (Fig. 1), with ordinal coordinates $(n_j, n_k, k_3, k_4, \ldots, k_N)$, $(n_j, n_k + 1, k_3, k_4, \ldots, k_N)$, $(n_j + 1, n_k, k_3, k_4, \ldots, k_N)$, $(n_j + 1, n_k + 1, k_3, k_4, \ldots, k_N)$. The $N - 2$ coordinates $(k_3, k_4, \ldots, k_N)$ are the same for the four regions and correspond to the ordinal coordinates with respect to the grid lines generated by the vectors different from $\boldsymbol{e}_j$ and $\boldsymbol{e}_k$. These coordinates can be obtained by taking the lowest integer part of the projection of $\boldsymbol{x}$ (for the given j and k) along each star vector. Thus, for the direction $\boldsymbol{e}_l$, and taking into account the shift terms, for the given j and k, the corresponding ordinal coordinates are

$$\begin{aligned} k_l &= \lfloor \boldsymbol{x} \cdot \boldsymbol{e}_l - \alpha_l \rfloor + 1 \\ &= \lfloor (x_{n_j} \boldsymbol{u}_{jk} + x_{n_k} \boldsymbol{u}_{kj}) \cdot \boldsymbol{e}_l - \alpha_l \rfloor + 1 \,, \end{aligned} \tag{7}$$

where $\lfloor \; \rfloor$ is the lowest integer function and the presence of 1 is due to the way in which one is labeling the space between grids.

Equations (7) and (3) imply that one of the vertices associated with the intersection of the lines j and k, is given by the dual transformation

$$\begin{aligned} \boldsymbol{t}^0_{n_j, n_k} = {} & n_j \boldsymbol{e}_j + n_k \boldsymbol{e}_k \\ & + \sum_{l \neq j \neq k} (\lfloor (x_{n_j} \boldsymbol{u}_{jk} + x_{n_k} \boldsymbol{u}_{kj}) \cdot \boldsymbol{e}_l - \alpha_l \rfloor + 1) \boldsymbol{e}_l \,. \end{aligned}$$

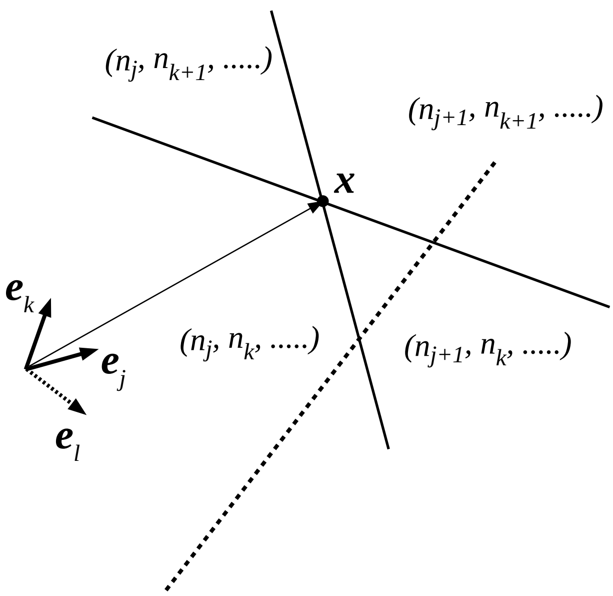

Fig. 1. The neighborhood of the intersection point of two lines perpendicular to the vectors $\boldsymbol{e}_j$ and $\boldsymbol{e}_k$.

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By considering the $\binom{N}{2}$ pairs (jk), one can obtain the vertex associated to each pair. Using the definition for a_{ij}, the complete set is written as

$$t^0_{n_j,n_k} = \sum_{k<j}^{N} \left(n_j \boldsymbol{e}_j + n_k \boldsymbol{e}_k + \sum_{l \neq j \neq k} \left(\left\lfloor x_{n_j} \frac{a_{jl}}{a_{jk}} + x_{n_k} \frac{a_{kl}}{a_{jk}} - \alpha_l \right\rfloor + 1 \right) \boldsymbol{e}_l \right). \tag{8}$$

The remaining three vertices of the rhombus associated to each pair (jk) have coordinates:

$$\begin{aligned} t^1_{n_j,n_k} &= t^0_{n_j,n_k} - \boldsymbol{e}_j, \\ t^2_{n_j,n_k} &= t^0_{n_j,n_k} - \boldsymbol{e}_k, \\ t^3_{n_j,n_k} &= t^0_{n_j,n_k} - \boldsymbol{e}_j - \boldsymbol{e}_k . \end{aligned} \tag{9}$$

A similar reasoning applies to multigrids in $\mathbb{R}^3$. Here we have to take into account that the intersection of the planes (sjk) defines eight regions whose ordinal coordinates generate the eight vertices of a rhombohedron by means of the dual transformation. The expression for the intersection of these planes is similar, and the coordinates of the vertex of the dual rhombohedron associated to the planes (sjk) is given by

$$t^0_{n_s,n_j,n_k} = n_s \boldsymbol{e}_s + n_j \boldsymbol{e}_j + n_k \boldsymbol{e}_k + \sum_{l \neq j \neq k \neq s} \left(\left\lfloor x_{n_s} \frac{V_{lsj}}{V_{sjk}} + x_{n_j} \frac{V_{ljk}}{V_{sjk}} + x_{n_k} \frac{V_{slk}}{V_{sjk}} - \alpha_l \right\rfloor + 1 \right) \boldsymbol{e}_l .$$

By considering the $\binom{N}{3}$ triplets (sjk), the vertices associated to each triplet are

$$t^0_{n_s,n_j,n_k} = \sum_{k<j<s}^{N} \left(n_s \boldsymbol{e}_s + n_j \boldsymbol{e}_j + n_k \boldsymbol{e}_k + \sum_{l \neq j \neq k \neq s} \left(\left\lfloor x_{n_s} \frac{V_{lsj}}{V_{sjk}} + x_{n_j} \frac{V_{ljk}}{V_{sjk}} + x_{n_k} \frac{V_{slk}}{V_{sjk}} - \alpha_l \right\rfloor + 1 \right) \boldsymbol{e}_l \right). \tag{10}$$

As in the case of $\mathbb{R}^2$, the remaining seven vertices of the rhombohedron associated to each triplet (sjk) have coordinates:

$$\begin{aligned} t^1_{n_s,n_j,n_k} &= t^0_{n_s,n_j,n_k} - \boldsymbol{e}_s , \\ t^2_{n_s,n_j,n_k} &= t^0_{n_s,n_j,n_k} - \boldsymbol{e}_j , \\ t^3_{n_s,n_j,n_k} &= t^0_{n_s,n_j,n_k} - \boldsymbol{e}_k , \\ t^4_{n_s,n_j,n_k} &= t^0_{n_s,n_j,n_k} - \boldsymbol{e}_j - \boldsymbol{e}_k , \\ t^5_{n_s,n_j,n_k} &= t^0_{n_s,n_j,n_k} - \boldsymbol{e}_s - \boldsymbol{e}_j , \\ t^6_{n_s,n_j,n_k} &= t^0_{n_s,n_j,n_k} - \boldsymbol{e}_s - \boldsymbol{e}_k , \\ t^7_{n_s,n_j,n_k} &= t^0_{n_s,n_j,n_k} - \boldsymbol{e}_j - \boldsymbol{e}_k - \boldsymbol{e}_s . \end{aligned} \tag{11}$$

For periodically spaced multigrids, equations (8) and (9) provide analytical expressions for the vertices of a quasilattice in two dimensions and equations (10) and (11) determine the vertices of a three-dimensional quasilattice.

4. Quasiperiodic grids

One advantage of the GDM over the projection method is that it is also possible to use quasiperiodic grids, which are constructed by parallel lines spaced by a quasiperiodic sequence. The main difference with the previously studied case of periodic grids, is that the spacing of grid planes or lines along a direction s is given by the quasiperiodic sequence (2):

$$x_{n_s} = n_s + \alpha_s + \frac{1}{\varrho_s} \lfloor n_s \sigma_s + \beta_s \rfloor ,$$

where σ_i is an irrational number which controls the sequence and frequency of the two possible intervals between x_n and x_{n+1}. α_i and β_i are shifts of the grid with respect to zero, and ϱ_s determine the ratio of the two different lengths that appears in the sequence.

In this case the intersection of three planes (sjk) of the multigrid are the solutions of the system

$$\begin{aligned} \boldsymbol{x} \cdot \boldsymbol{e}_s &= x_{n_s} = n_s + \alpha_s + \frac{1}{\varrho_s} \lfloor n_s \sigma_s + \beta_s \rfloor , \\ \boldsymbol{x} \cdot \boldsymbol{e}_j &= x_{n_j} = n_j + \alpha_j + \frac{1}{\varrho_j} \left\lfloor n_j \sigma_j + \beta_j \right\rfloor , \\ \boldsymbol{x} \cdot \boldsymbol{e}_k &= x_{n_k} = n_k + \alpha_k + \frac{1}{\varrho_k} \lfloor n_k \sigma_k + \beta_k \rfloor , \end{aligned} \tag{12}$$

which, for the given s, j and k, are of the form

$$\boldsymbol{x} = x_{n_s} \boldsymbol{u}_{ksj} + x_{n_j} \boldsymbol{u}_{sjk} + x_{n_k} \boldsymbol{u}_{jks} .$$

To obtain the dual transformation, it is necessary to know its position in the grid with respect to each direction defined by the star vectors. In the periodic case it was only necessary to take the integer part of the scalar product between the vector pointing to the intersection point $\boldsymbol{x}$ and $\boldsymbol{e}_l$, since this product gives the projection of $\boldsymbol{x}$ in a direction perpendicular to the grid director vector, and the integer part gives the ordinal position in the grid. In the case of a quasiperiodic grid, it requires some more effort. Suppose that we want to obtain the ordinal position in the direction l. For a given s, j and k, define

$$u_l \equiv \boldsymbol{x} \cdot \boldsymbol{e}_l = x_{n_s} \frac{V_{lsj}}{V_{sjk}} + x_{n_j} \frac{V_{ljk}}{V_{sjk}} + x_{n_k} \frac{V_{slk}}{V_{sjk}} . \tag{13}$$

We are looking for an integer n_l such that,

$$x_{n_l} \le u_l \le x_{n_l+1} .$$

The goal is to calculate the ordinal position n_l if the distance x_{n_l} is known. This problem can be separated into two cases as is shown in the Appendix A, where the technical details are presented.

4.1 Case I: $\varrho_l + \sigma_l \ge 1 \;\; \forall\, l$

From equation (21) in the Appendix A it follows that the coordinates of the vertex of the dual rhombohedron associated to the planes (sjk) are given by

$$t^0_{n_s,n_j,n_k} = n_s \boldsymbol{e}_s + n_j \boldsymbol{e}_j + n_k \boldsymbol{e}_k + \sum_{l \neq s \neq j \neq k} \left(\left\lfloor r + \frac{1}{\varrho_l + \sigma_l} \{ (\lfloor r \rfloor + 1)\, \sigma_l + \beta_l \} \right\rfloor + 1 \right) \boldsymbol{e}_l ,$$

where

$$r = \frac{1}{1+(\sigma_l/\varrho_l)} \times \left(x_{n_s}\frac{V_{lsj}}{V_{sjk}} + x_{n_j}\frac{V_{ljk}}{V_{sjk}} + x_{n_k}\frac{V_{slk}}{V_{sjk}} - \alpha_l - (\beta_l/\varrho_l) \right),$$

and $\{\ \}$ is the fractional part function.

By considering the $\binom{N}{3}$ triplets (sjk), the set of vertices associated to each triplet is given by

$$\boldsymbol{t}^0_{n_s,n_j,n_k} = \sum_{k<j<s}^{N} \left(n_s\boldsymbol{e}_s + n_j\boldsymbol{e}_j + n_k\boldsymbol{e}_k + \sum_{l\neq s\neq j\neq k} \left(\left\lfloor r + \frac{1}{\varrho_l+\sigma_l}\{(\lfloor r\rfloor+1)\sigma_l+\beta_l\} \right\rfloor + 1 \right)\boldsymbol{e}_l \right). \tag{14}$$

4.2 Case II: $\varrho_l + \sigma_l < 1 \ \ \forall\, l$

Let us first write one of the vertices associated with the intersection of the planes (sjk) as

$$\boldsymbol{t}^0_{n_s,n_j,n_k} = n_s\boldsymbol{e}_s + n_j\boldsymbol{e}_j + n_k\boldsymbol{e}_k + \sum_{l\neq s\neq j\neq k} n_l\boldsymbol{e}_l\,,$$

In Appendix A for this case, there are two different expressions for the ordinal coefficients n_l:

1. If $\left\{\frac{u_l-\alpha_l}{T_l}\right\}T_l \geq 1/\sigma_l$, where $T_l = (1/\sigma_l) + (1/\varrho_l)$, then

$$n_l = \left(\left\lfloor \left(\left\lfloor \left\lfloor \frac{u_l-\alpha_l}{T_l} \right\rfloor \frac{1}{\sigma_l} \right\rfloor + 1 \right) \frac{1}{\sigma_l} \right\rfloor + 1 \right). \tag{15}$$

An exception occurs when $\lfloor (u_l-\alpha_l)/T_l \rfloor = -1$; in that case $n_l = 0$.

2. If $\left\{\frac{u_l-\alpha_l}{T_l}\right\}T_l < 1/\sigma_l$, then the corresponding coefficient is

$$n_l = \left\lfloor u_l - \alpha_l - \frac{1}{\varrho}\left(\left\lfloor \left\lfloor \left\lfloor \frac{u_l-\alpha_l}{T_l} \right\rfloor \frac{1}{\sigma_l} \right\rfloor \sigma \right\rfloor + 1 \right) \right\rfloor. \tag{16}$$

An exception occurs as well when $\lfloor (u_l-\alpha_l)/T_l \rfloor = 0$; in that case $n_l = \lfloor u_l - \alpha_l \rfloor$.

The set of vertices associated to each of the $\binom{N}{3}$ triplets (sjk), is then given by

$$\boldsymbol{t}^0_{n_s,n_j,n_k} = \sum_{k<j<s}^{N} \left(n_s\boldsymbol{e}_s + n_j\boldsymbol{e}_j + n_k\boldsymbol{e}_k + \sum_{l\neq s\neq j\neq k} n_l\boldsymbol{e}_l \right). \tag{17}$$

The values of n_l are obtained from the previous cases (i) or (ii), and u_l is defined in (13).

Depending on the values of $\varrho_l + \sigma_l$, equations (14–17) provide analytical expression for the vertex coordinates of quasilattices obtained from a quasiperiodically spaced grid. As in the case of periodic multigrids, the remaining vertices of the rhombohedron associated to each triplet (sjk) are obtained from (11). The reduction to the case of multigrids in 2D is straightforward. In Fig. 2, two examples of quasiperiodic tilings obtained using quasiperiodic grids, and the above formulae, are shown.[1]

[1] A *Mathematica* notebook for generating lattices from periodic of quasiperiodic grids, using the derived formulae, is available under request. Contact G. Naumis: naumis@fisica.unam.mx

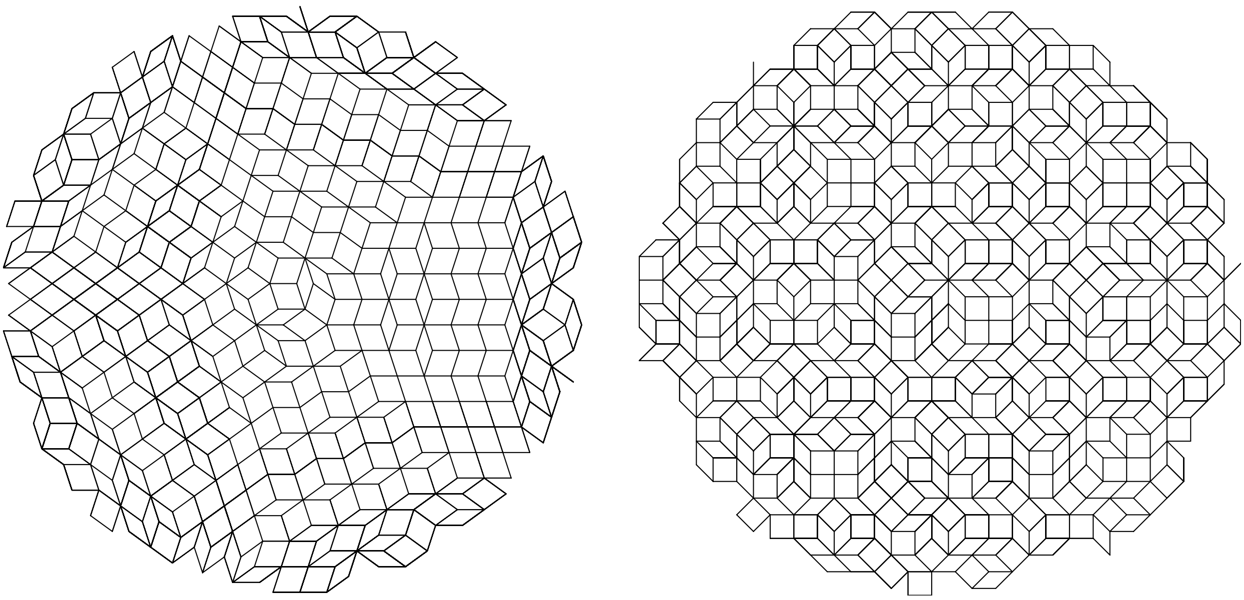

Fig. 2. Examples of pentagonal (left) and octagonal (right) quasiperiodic tilings obtained by using quasiperiodic grids. For the pentagonal tiling, a star of vectors pointing to the vertices of a regular pentagon was used, and parameter values are
$\boldsymbol{\alpha} = (-0.1, 0.3, -0.3, 0.2, 0.0)$,
$\boldsymbol{\beta} = (0.0, 0.0, 0.0, 0.0, 0.0)$,
$\boldsymbol{\sigma} = (3^{-3/2}, 3^{-3/2}, 3^{-3/2}, 3^{-3/2}, 3^{-3/2})$ and
$\boldsymbol{\varrho} = (2^{-5/2}, 2^{-5/2}, 2^{-5/2}, 2^{-5/2}, 2^{-5/2})$.
Thus $\varrho_l + \sigma_l = 0.369 < 1 \ \forall\, l$, and equations (21–23) apply. The octagonal tiling was obtained with a star of four vectors pointing to four vertices of a regular octagon and parameter values
$\boldsymbol{\alpha} = (0.000001, 0.000001, 0.000001, 0.000001)$,
$\boldsymbol{\beta} = (2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2})$,
$\boldsymbol{\sigma} = (1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$ and
$\boldsymbol{\varrho} = (8^{-1/2}, 8^{-1/2}, 8^{-1/2}, 8^{-1/2},)$.
In this case $\varrho_l + \sigma_l = 1.06 \leq 1 \ \forall\, l$, and formula (20) applies

5. Conclusions

Using the generalized dual method to generate quasiperiodic structures in two and three dimensions, we obtain closed analytical expressions for the vertex coordinates of a quasiperiodic lattice derived by using periodic or quasiperiodic grids. These formulae offers an efficient and general way to generate quasiperiodic structures and can be useful to perform analytical and systematic calculations in these structures, related with physical properties of quasicrystals.

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Appendix

Ordinal positions for quasiperiodically spaced multigrids

Given a number u, we need to find an integer n such that,

$$x_n \le u < x_{n+1}\,. \tag{18}$$

As a first step, we will find n if $u = x_n$. Using the expression for x_n and the identity $x = \lfloor x \rfloor + \{x\}$, where $\{\ \}$ is the fractional part function, we obtain,

$$n = \left\lfloor \frac{x_n - \alpha - (\beta/\varrho)}{1 + (\sigma/\varrho)} \right\rfloor + \left\{ \frac{x_n - \alpha - (\beta/\varrho)}{1 + (\sigma/\varrho)} \right\} + \frac{1}{\varrho + \sigma} \{n\sigma + \beta\}\,. \tag{19}$$

If we compare the integer parts on the left and right sides of the last equation, we can distinguish two cases, $\varrho_l + \sigma_l \ge 1$ (case I) and $\varrho_l + \sigma_l < 1$ (case II). In case I, the last term is always less than one, and can be treated in a simple way. Case II is more complicated and requires a different approach.

Case I: $\varrho_l + \sigma_l \ge 1 \ \ \forall\, l$

Since $\varrho + \sigma \ge 1$ and the last term in equation (19) is always less than one, one may think that the following equation must hold,

$$n = \lfloor y \rfloor\,, \quad \text{where} \quad y = \frac{x_n - \alpha - (\beta/\varrho)}{1 + (\sigma/\varrho)}\,.$$

However, it may happen that the sum of the fractional parts is more than one. In such a case,

$$n = \lfloor y \rfloor + 1\,.$$

To solve this problem, we take the integer part on both sides of equation (19) to get,

$$n = \left\lfloor y + \frac{1}{\varrho + \sigma} \{n\sigma + \beta\} \right\rfloor \tag{20}$$

Observe that n appears in both sides of the equation. It is easy to show that if $n = \lfloor y \rfloor$ then the fractional part in the above equation has no effect. This fractional part is relevant provided that $\lfloor y \rfloor + 1$. This allows us to use the formula in a recursive way, that is, we put $n = \lfloor y \rfloor + 1$ on the right-hand side of the equation to obtain the final expression for n in terms of x_n,

$$n = \left\lfloor y + \frac{1}{\varrho + \sigma} \{(\lfloor y \rfloor + 1)\,\sigma + \beta\} \right\rfloor .$$

Since x_n is always a growing function, it is clear that for any d the same relation is hold:

$$n = \left\lfloor r + \frac{1}{\varrho + \sigma} \{(\lfloor r \rfloor + 1)\,\sigma + \beta\} \right\rfloor , \tag{21}$$

where

$$r = \frac{u - \alpha - (\beta/\varrho)}{1 + (\sigma/\varrho)}\,.$$

Case II: $\varrho_l + \sigma_l < 1 \ \ \forall\, l$

Since the fractional part in the last part of equation (19) is in this case larger than one, this equation is not useful and we proceed in a very different way. Let us first consider the function,

$$f(z) \equiv x(z) - \alpha = z + \frac{1}{\varrho} \lfloor z\sigma \rfloor\,,$$

where z is a continuous variable, and define

$$w = u - \alpha\,.$$

The goal is to find an integer n such that

$$f(n) \le w < f(n+1)\,.$$

Observe that for simplicity we have omitted β since its only effect is to shift the entire sequence, and its inclusion can be treated in a similar way. A plot of the function $f(z)$ is shown in Fig. 3. As we can see, $f(z)$ has the following property,

$$f\left(z + \frac{1}{\sigma}\right) = f(z) + T\,, \quad \text{where} \quad T = \frac{1}{\sigma} + \frac{1}{\varrho}\,.$$

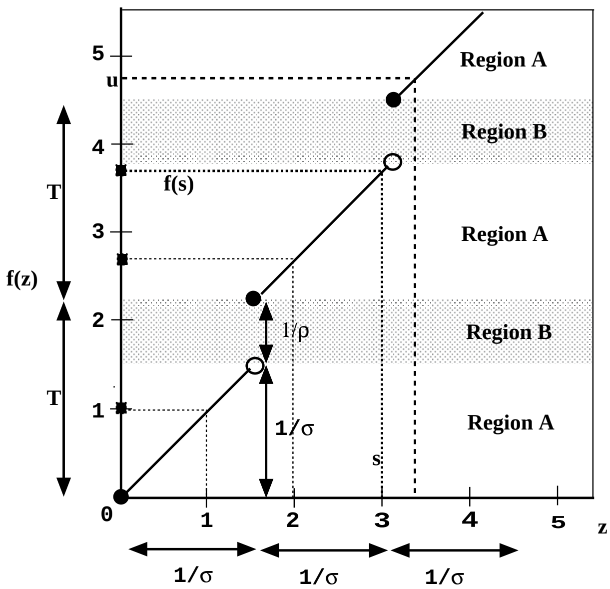

Fig. 3. Plot of the auxiliary function $f(z)$ used to find the ordinal coordinates when $\varrho_l + \sigma_l < 1$ (case II). In this case, $\alpha = 0$ and $u = w$.

T can then be interpreted as a sort of average period of the sequence. To obtain n we again divide the computation into two steps. First we calculate how many complete periods T can we put in w (let t be this number) and the corresponding number of integers in the z axis that is possible to have in the distance t/σ. Then, we obtain the number of integers contained in the interval $\Delta z = f^{-1}(w) - f^{-1}(tT)$, that corresponds to the distance between the last complete average period and w.

For the first goal, we observe that for a given u, the number of complete periods T contained in the interval is,

$$t = \left\lfloor \frac{w}{T} \right\rfloor,$$

and the corresponding integers that we can put in the z axis, for this number of periods, is t/σ. Now, let us define the integer

$$s \equiv \left\lfloor \left\lfloor \frac{w}{T} \right\rfloor \frac{1}{\sigma} \right\rfloor.$$

This number is the greatest integer such that $f(s) < tT$. For example, $s = 3$ in Fig. 3, since it is the greatest integer that completes an integer number of periods (in this case $t = 2$) before $d = 4.8$.

Next we consider how many integers are contained in the interval Δz. To do this, we first compute the distance in the $f(z)$ axis between w and tT, and we call this distance Δf. Such a number is obtained in a simple way from

$$\Delta f = \left\{ \frac{w}{T} \right\} T.$$

and here we again have two cases.

From Fig. 3 it is clear that if $\Delta f \geq 1/\sigma$ (regions A), no more that $\lfloor 1/\sigma \rfloor$ integers in the z axis can be put in the interval Δz. From here we obtain that the corresponding n, for a given u, is

$$n = s + \lfloor 1/\sigma \rfloor = \left\lfloor (s+1) \frac{1}{\sigma} \right\rfloor.$$

An exception occurs when $\Delta f \geq 1/\sigma$ and $\lfloor w/T \rfloor = -1$; in this case $n = -1 \neq 0$. This exception is a consequence of the fact that near $w = 0$ there is no complete period.

When $\Delta f < 1/\sigma$ (regions B), the number of integers is obtained by calculating the distance between w and $f(s) + 1/\sigma$. The number of integers m that corresponds to such interval is

$$m = \lfloor w - (f(s) + 1/\sigma) \rfloor.$$

Finally, the integer n is given by,

$$n = s + m = \left\lfloor w - \frac{1}{\varrho} (\lfloor s\sigma \rfloor + 1) \right\rfloor.$$

If $\lfloor w/T \rfloor = 0$ a new exception occurs and in this case $s = -1$.

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