

The stability of the renormalization group as a diagnostic tool for localization and its application to the Fibonacci case

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Abstract

A discussion of the investigation of electronic localization at the thermodynamical limit is given; we show that the usual methods involve practical difficulties, especially for quasicrystals. A method based on the scaling of bands of a supercrystal is proposed. Then, the nature of the localization is related to the stability of the renormalization group around the fixed points of a trace map. For quasiperiodic systems, the scaling exponents are obtained using modified Lyapunov exponents and a Thouless formula. The case of the Fibonacci chain is considered as an example, and analytical expressions for the scaling exponents are found.

1. Introduction

During the last 30 years, the renormalization group [1] has been used as a powerful tool in the study of localization–delocalization transitions in different contexts—for example, in the free-volume percolation model of the glass transition [2], and in the localization scaling theory [3], where the Anderson transition from extended to localized states was recognized by finding the fixed point of the renormalization group [2]. However, the relationship between the stability of the renormalization group and localization has not been fully explored. For example, one can ask whether it is possible to obtain the localization length (ξ) of a state from the renormalization technique. This question is relevant, since in one dimension (1D), physicists calculate ξ numerically as the inverse of the Lyapunov exponent (LE) of a transfer matrix norm [4, 5]. From a mathematical point of view, the LE involves right- and left-limit procedures at the localization centre [6]; however, in many works, only the norm of the transfer matrix is taken [4, 5]. As a result, the problem of the match between the right and left solutions is not considered (this corresponds to the well known Borland paradox [6]). Furthermore, in the sense used by many people, the LE of an eigenenergy tends to zero in the thermodynamical limit (as we will discuss later), regardless of the wavefunction shape. Thus, the method does not produce the required value of ξ . Although this is a problem, it has not been seriously taken into account, and it is clear that the present situation is not satisfactory. In 1D, the correct path

[See endnote 1](#)

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to follow to obtain ξ is to find the localization centre of the wavefunction, and then proceed to obtaining the LE using limits from the left and right sides of a 1D lattice [7, 8]. The problem is how to find the localization centre and to extend the results to higher dimensions.

A clear demonstration of the inherent difficulties of this problem has been given for quasiperiodic systems. Since the discovery of quasicrystals [11], which are non-periodic solids with long-range orientational order, the study of quasiperiodic Hamiltonians has been a very active field of research. In 1D, the Fibonacci chain and the Harper model are the most studied systems [13–17]. For the Fibonacci chain, it has been proved that the spectrum is singular continuous (fractal), and the wavefunctions are critical [18]. For 2D and 3D, the nature of the spectrum and the localization properties are still open questions, although it has been shown [19] that the topological disorder and the associated frustration of the wavefunction make the problem very different from the 1D case. The renormalization group has been a very useful tool in studying the phonon and electronic spectra of quasiperiodic systems in 1D [13, 20] and 2D [19, 21], due to the self-similar nature of the potential that makes this formalism particularly involved. However, even in 1D it has not been possible to find the localization length and the scaling properties for all the wavefunctions, and the roles of the boundary conditions in the thermodynamical limits are not well understood.

Hence, it is clear that the issue of how one should take the thermodynamical limit to study localization is delicate, and that its importance has been underestimated in the literature. An alternative approach for investigating localization has been proposed by the present author in [22], adapting arguments due to Thouless [9] and Sire [10]. In 1D, this approach turned out to be closely connected with the nature of the transfer matrix trace map renormalization group flow; thus it can be used as a diagnostic indicator for localization [12]. However, in [22], various points were not clear, since the relationship between localization and stability was obtained using an approximation, and the nature of the boundary conditions in the thermodynamical limit was not discussed. Also, the theory was derived only for the ‘on-site’ (also called ‘diagonal’) problem [22].

The aim of this paper is twofold. On one hand, we aim to provide a solid basis for the previous work [22] by showing how the boundary conditions at the thermodynamical limit, the scaling of bands, and the method of the transfer matrix are interconnected, and to give a warning regarding how to take the thermodynamical limit in the localization problem. This new unified point of view is able to produce expressions for the LE in terms of the trace map, without using approximations. As an example, a Thouless type of formula for scaling exponents in quasicrystalline systems is derived. Also, generalizations to the ‘diagonal’ and ‘mixed’ Hamiltonian problems are presented. On the other hand, as an application of the theory, we present new analytical results for the scaling exponents and the nature of localization in a quasiperiodic system, using the Kohmoto, Kadanoff, and Tao [13] recurrence relation for the trace of a Fibonacci chain. These results can be extended to other quasiperiodic [18, 20, 23] and disordered potentials.

The work is arranged as follows. In section 2 a discussion of the boundary conditions and the thermodynamical limits is given, and the method for using the renormalization group is presented; it is based on the scaling of bands. In section 3 the method is applied to the case of the Fibonacci chain, and in section 4 the conclusions are given.

2. Scaling of bands and localization

In this section, we will discuss the need for a new tool, derived from the transfer matrix, for investigating localization. Let us start the discussion by giving some general reflections on the abuse of LEs in the literature.

[See endnote 2](#)

As a model, we use a very simple Hamiltonian that contains a lot of the physics involved in the localization problem: a 1D s-band tight-binding Hamiltonian. In more dimensions, the method of evaluating localization using the scaling of bands works in a similar way. The Hamiltonian is defined on a chain of n sites, with an on-site potential V_n at site n , and the hopping integral t_{n+1} , for hopping between sites n and $n + 1$. The corresponding Schrödinger equation for this model is

$$t_n \psi_{n-1} + t_{n+1} \psi_{n+1} + V_n \psi_n = E \psi_n, \quad (1)$$

where ψ_n is the value of the wavefunction at site n . The usual procedure used to investigate localization in this system is as follows. Equation (1) can be rewritten in terms of the transfer matrix $M(n, E)$ and a vector Ψ_n with components (ψ_n, ψ_{n-1}) , that satisfies

$$\Psi_{n+1} \equiv \begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \begin{pmatrix} \frac{E-V_n}{t_{n+1}} & -\frac{t_n}{t_{n+1}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} \equiv M(n, E) \Psi_n. \quad (2)$$

The wavefunction at site n is given by successive applications of equation (2):

$$\Psi_n = M(n, E) M(n-1, E) \cdots M(1, E) \Psi_1 \equiv T(n, E) \Psi_1. \quad (3)$$

The spectrum is the set of energies for which the trace norm $\tau_n(E) \equiv \text{tr } T(n, E)$ is less than 2 [18]. This comes from the following line of thought. The LE that measures the growth of the wavefunction is [24]

$$\gamma(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|T(n, E)\| = \lim_{n \rightarrow \infty} \frac{1}{n} \ln |\lambda_{\max}|, \quad (4)$$

where λ_{\max} is the greatest eigenvalue of $T(n, E)$. λ_{\max} can be found by using the characteristic equation of the transfer matrix, and the trace invariance under unitary transformations. The two eigenvalues of $T(n)$ are

$$\lambda_{\pm}(E) = \frac{\tau_n(E) \pm \sqrt{\tau_n^2(E) - 4 \det T(n, E)}}{2} \quad (5)$$

where $\det T(n, E)$ is the determinant of the matrix $T(n, E)$. Using the fact that the determinant of a product is the product of determinants, we get

$$\det T(n, E) = \prod_{j=1}^n \det M(j, E) = \frac{t_1 t_2 \cdots t_{n-1} t_n}{t_2 t_3 \cdots t_n t_{n+1}} = \frac{t_1}{t_{n+1}}.$$

For a periodic crystal, or if we use cyclic boundary conditions, $t_1 = t_{n+1}$, and $\det T(n, E) = 1$. Equation (5) reads

$$\lambda_{\pm}(E) = \frac{\tau_n(E) \pm \sqrt{\tau_n^2(E) - 4}}{2}. \quad (6)$$

In an energy that belongs to the spectrum, $\lambda_+(E)$ and $\lambda_-(E)$ are complex numbers [6], both with unitary norm. This means that the LE at the corresponding energy is zero when $n \rightarrow \infty$ since $\|T(n, E)\|$ is always 1. These energies correspond to solutions that satisfy the boundary conditions, i.e., solutions that are bounded at infinity. From equation (6), it follows that for infinite crystals or cyclic boundary conditions, E must satisfy $|\tau_n(E)| \leq 2$ in order to be an allowed energy. This is a necessary but not sufficient condition. To further investigate this condition, we will consider the difference between an infinite crystal and cyclic boundary conditions.

Suppose for the moment that we are dealing with a finite cell of size n that can be the unitary cell of an *infinite crystal*. In this infinite crystal, the boundary conditions imply that $\psi_{n+2} = e^{i(n+1)k(E)} \psi_1$ and $\psi_{n+1} = e^{i(n+1)k(E)} \psi_0$, where $e^{i(n+1)k(E)}$ is a phase and $k(E)$ the usual wavevector. The same cell can be used to make a chain with *cyclic boundary conditions*, where

$\psi_{n+2} = \psi_1$ and $\psi_{n+1} = \psi_0$, which corresponds to setting $k(E) = 0$ in the boundary conditions of the infinite crystal. For this crystal, from equation (2) we get

$$\begin{pmatrix} \psi_{n+2} \\ \psi_{n+1} \end{pmatrix} = e^{i(n+1)k(E)} \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = T(n, E) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}, \quad (7)$$

and for having a solution for non-trivial values of ψ_1 and ψ_0 ,

$$\det(T(n, E) - e^{i(n+1)k(E)} I) = 0.$$

This condition is satisfied only when

$$\tau_n(E) = 2 \cos((n+1)k(E)). \quad (8)$$

Since the cosine takes values between -1 and 1 , all energies that satisfy $|\tau_n(E)| \leq 2$ are in the spectrum. For cyclic boundary conditions, $k(E) = 0$, and thus only energies where $\tau_n(E) = 2$ are eigenvalues. Notice that the allowed energies are exactly the same as in an infinite crystal but with the restriction $\tau_n(E) = 2$. This very important fact will be used later, and provides great insight into how to use the transfer matrix in order to detect localized states, since for cyclic boundary conditions nothing was said about the localization nature of the states.

When equation (8) holds, from equations (4) and (5) the LE is always zero and this proves that, in general, the condition $|\tau_n(E)| \leq 2$ cannot be used in conjunction with equation (4) at the thermodynamical limit, as is usually done. Now the question is how to use the trace to detect a solution that corresponds to a localized state or a critical state, i.e., what the equivalent for the condition $|\tau_n(E)| \leq 2$ is. Although the same condition has been used for many systems, such as quasicrystals [13], it is not clear whether the condition holds or not and, usually, some assumptions are taken for granted, rather than discussed. Our purpose here is to clarify this essential point.

The solution is to note that the allowed energies always satisfy $\tau_n(E) = 2$, irrespective of the nature of the state, when cyclic boundary conditions are used. Thus, this can be used as a condition that holds even for localized states. However, we can make use of the fact that this is just an energy inside the spectrum of the infinite crystal, and thus use the condition $|\tau_n(E)| \leq 2$ and take the limit $n \rightarrow \infty$ to detect an eigenstate. How is this possible? In the condition $|\tau_n(E)| \leq 2$, what we are really doing is constructing a supercrystal, where the unit cell is given by a piece of the chain to be considered. This assumption has been made in the study of quasicrystals, where the procedure is known as ‘rational approximation’. Hence, the condition $|\tau_n(E)| \leq 2$ can be used even for a quasiperiodic potential [13]. But if we use this condition, again we cannot detect localization from the usual procedure, equation (4).

The big advantage of using the supercrystal is that, now, the issue of localization can be related to the scaling of bands, adapting arguments from Thouless [9] and Sire [10], as was done in [22]. The idea is to build a supercrystal, by considering a piece of chain with n sites as the unit cell; the cell can be amorphous, quasiperiodic, or crystalline. The resultant supercrystal has Bloch solutions and a spectrum with n bands, where each bandwidth W_n depends on the overlap of the wavefunctions at the interfaces of contiguous cells [22]. Now, we can relate localization to the stability of the trace, since W_n depends on the trace through the band edges (energies that we denote as E_s). The band edges are the points where

$$\tau(E_s) = \pm 2 \quad \text{and} \quad \left. \frac{d\tau_n(E)}{dE} \right|_{E=E_s} \neq 0.$$

The condition on the derivative is due to the fact that in a band edge the trace must cross the line defined by $\tau(E_s) = \pm 2$ (see figure 1). In [22] an approximate derivation for the numerical relationship between stability and localization was found. Here, we present an exact derivation

[See endnote 3](#)

[See endnote 4](#)

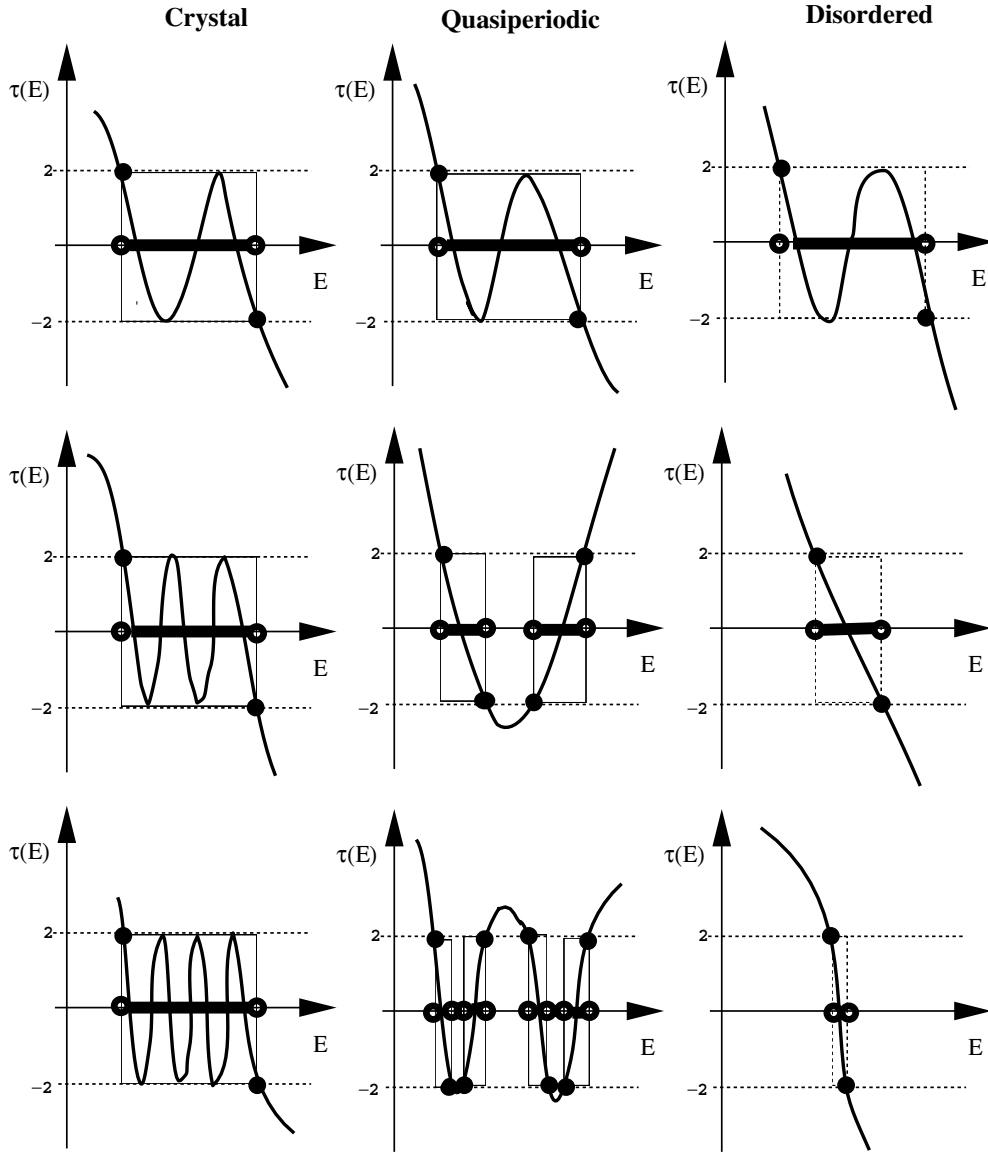


Figure 1. The evolution of the trace for different sizes of the system for three kinds of localization. The bands edges E_s , where $\tau_n(E_s) = \pm 2$, are shown as circles, while the corresponding traces are shown as dark circles. The bandwidths are the dark lines on the E -axis. Notice that it is possible to have oscillations of the trace inside the spectrum, as happens for example in the case of a monocrystal, where the number of oscillations is equal to the number of sites. The unit cell sizes are, from top to bottom, $N = 1, 2$, and 4 sites.

using the Thouless formula [6]. We proceed as follows. First, we write the trace of the infinite supercrystal as a polynomial:

$$\tau_n(E) - 2 = C_n \prod_{i=1}^n (E - E_i), \quad (9)$$

where E_i are the roots of $\tau_n(E) - 2 = 0$, and C_n is a constant. After taking the logarithm of

the derivative with respect to E evaluated at one of the band edges (E_s), we get

[See endnote 5](#)

$$\ln\left(\frac{d\tau_n(E)}{dE}\right)_{E=E_s} = \sum_{i \neq s} \ln(E_s - E_i). \quad (10)$$

Observe that the derivative in a band edge is never zero, so there are no problems of having a divergence. Now, we make the following observation. Since the E_i are the points where $\tau_n(E) = 2$, these are the only values that satisfy a cyclic boundary condition for the same finite cell of size n , as was shown before. Taking into account this observation and that each state has a weight $1/n$ in the density of states, we can perform the sum in equation (10) by using the density of states ($\rho_n(E)$) of the cyclic problem. This leads to

$$\frac{1}{n} \ln\left(\frac{d\tau_n(E)}{dE}\right)_{E=E_s} = \int_{-\infty}^{\infty} \rho_n(E') \ln(E_s - E') dE'. \quad (11)$$

Using the well known Thouless formula [6], which stipulates that the inverse of the localization length $\xi(E_s)$ is equal to the integral that appears in the previous equation, and taking the thermodynamical limit $n \rightarrow \infty$, we get an expression that now depends on the trace:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln\left(\frac{d\tau_n(E)}{dE}\right) = \frac{1}{\xi(E)}.$$

Notice that we can drop the 's', since it is understood that the derivative must be evaluated at a band edge. The relation to the stability of the trace map is evident. Now, it corresponds to a LE, used in the sense that gives how much two trajectories diverge under the action of $\tau_n(E)$, when the initial conditions differ by a small quantity.

For quasiperiodic systems, the wavefunction decays as $\psi_n \approx n^{\beta_n(E)}$ [18]; hence the LE can be zero. To avoid this problem, instead of dividing by n in equation (11), we divide by $2 \ln n$, since for the supercrystal, $W_n \simeq \langle \psi_n | H | \psi_1 \rangle \simeq n^{2\beta_n(E)}$. Then we get a modified LE that gives the scaling exponent of the wavefunction, and a Thouless formula that is appropriate for quasiperiodic systems:

$$\beta_n(E) \equiv \frac{1}{2 \ln n} \ln\left(\frac{d\tau_n(E)}{dE}\right) = \frac{n}{\ln n} \int_{-\infty}^{\infty} \rho_n(E') \ln(E - E') dE'. \quad (12)$$

In general, the scaling exponent obtained from this equation is a minimum, as one can argue from the fact that the overlap between cells is dominated by the maximal overlap. From a physical point of view, this is the most important exponent, because it rules the maximum conductivity of the system. In the cases where the gaps between the bands are very small, some corrections may appear due to hybridization of levels in neighbouring cells.

For many systems, there are recurrence relations for the trace of a certain chain size [13] as a function of the trace of m smaller chains of lengths l_j :

$$\tau_n(E) = f(\tau_{l_1}(E), \tau_{l_2}(E), \dots, \tau_{l_m}(E)). \quad (13)$$

These types of relation are called trace maps, and they are powerful tools for studying the spectrum, especially for quasicrystals. The properties of localization are determined by the stability of the trace map around the points $\tau_n = \pm 2$, since, for a localized state, the band shrinks in an exponential way as the system grows, due to a decreasing overlap between neighbouring cells [22]. Thus, a localized state corresponds to a repulsive fixed point of the trace map, as shown in figure 1. For extended states, the band edges do not change with the system size because the overlap is size independent, and $\tau_n(E_s) = \pm 2$ for any n and fixed E_s . $\tau_n(E_s) = \pm 2$ is a fixed point of the trace map, and equation (13) must satisfy

$$\pm 2 = f(\pm 2, \pm 2, \pm 2, \dots).$$

The point $\tau_n(E_s) = \pm 2$ is hyperbolic in nature, since, as the system is scaled, the trace for energies inside the band remains bounded, while for other energies it goes to infinity. In 1D quasiperiodic systems, the bands are subdivided as n grows in a similar way to in the generation of a Cantor set, and thus the number of points E_s grows with the system size (see figure 1).

3. Application for a quasiperiodic potential: the Fibonacci lattice

As an example of the use of the technique proposed, we will obtain the analytic scaling exponents for the wavefunctions of the simplest quasiperiodic potential: the Fibonacci chain. For this system it has been rigorously proved that the LE in an energy inside the spectrum is zero [18, 27]. Although now there are many well developed mathematical techniques for studying quasiperiodic Hamiltonians [28–30], the problem of determining the scaling exponents of the wavefunction has been approached from numerical calculations. Only for a certain state is the exact scaling exponent known, and some discrepancies in its value appear in the literature [26, 32]. The objective of this section is to apply the technique proposed to show how it works, and to obtain an independent deduction of the nature of the spectral properties. The method is able to produce scaling exponents of the wavefunctions at all energies, and it reproduces the correct analytical exponent for the only energy for which an analytical exponent is known.

We start by defining the model. In the diagonal problem, the potential $V(n)$ is taken from a sequence of two letters, A and B. The sequence is generated using the following recursive rule: $A \rightarrow B$ and $B \rightarrow BA$. It has been proved that the trace of a chain with length $F(l)$ ($F(l)$ is the Fibonacci number of generation l) is given by [13]

$$x_l(E) = x_{l-1}(E)x_{l-2}(E) - x_{l-3}(E), \quad (14)$$

where x_l is half the trace ($x_l = \tau_l(E)/2$). The map has as initial conditions $x_{-1}(E) = 1$, $x_0(E) = (E + \lambda)/2$, and $x_1(E) = (E + \lambda)/2$, where $\lambda = |(V_A - V_B)|/2$. The nature of the states is easily obtained from the approach presented in this work. First one needs to obtain the fixed points of the map; in this case, if we denote the fixed point as x^* , from equation (14),

$$x^* = 2(x^*)^2 - x^*,$$

two solutions are found: $x^* = 0$ and 1 (notice that this corresponds to $\tau(E) = 2$). However, 0 is not consistent with the initial conditions, since the map has an invariant [13]:

$$x_{l+1}^2(E) + x_l^2(E) + x_{l-1}^2(E) - 2x_{l+1}(E)x_l(E)x_{l-1}(E) = \lambda^2 + 1.$$

From the invariant, is easy to show that the other fixed point ($x^* = 1$) occurs only when we have a periodic chain, i.e., $\lambda = 0$ and $V(n) = V_A = V_B$. For this value of the parameter λ , the system presents extended states, but it is instructive to apply the method presented to determine localization. In principle, from the fact that we have fixed points of the map for $\lambda = 0$, this means that the states are localized or extended. To go further, we need to investigate the nature of the flux of the renormalization group around $x^*(E) = 1$. Stability is obtained by carrying out a linear stability analysis. We use that the map can be seen as a trajectory in 3D [13], where the coordinates of a point $\mathbf{p} = (x_{n+1}, x_n, x_{n-1})$ are iterated to form a 3D map:

$$\begin{aligned} x_{n+2}(\mathbf{p}) &= x_{n+1}x_l - x_{l-1} \\ x_{n+1}(\mathbf{p}) &= x_l \\ x_n(\mathbf{p}) &= x_{l-1}. \end{aligned}$$

The analysis consists in a Taylor expansion around the fixed point. Since the map is a vectorial function, the derivative is a 3×3 matrix, and if we make a move around the fixed

point given by the small vector $d\mathbf{p} \equiv (\varepsilon_x, \varepsilon_y, \varepsilon_z)$, the expansion is

$$\begin{pmatrix} x_{n+2}(\mathbf{p} + d\mathbf{p}) \\ x_{n+1}(\mathbf{p} + d\mathbf{p}) \\ x_n(\mathbf{p} + d\mathbf{p}) \end{pmatrix} \approx \begin{pmatrix} x_{n+2}(\mathbf{p}) \\ x_{n+1}(\mathbf{p}) \\ x_n(\mathbf{p}) \end{pmatrix} + \begin{pmatrix} 2y_l & 2x_l & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix}.$$

The nature of the flux is obtained from the three eigenvalues of the derivative matrix [31] evaluated at the point $\mathbf{p} = (1, 1, 1)$; these eigenvalues are

$$r_1 = -1, \quad r_2 = \sigma^2, \quad r_3 = 1/\sigma^2,$$

where σ is the golden mean $(\sqrt{5} + 1)/2$. Since we have an eigenvalue bigger than one, and the other is less than one, the point is classified as hyperbolic; in one direction the flux diverges, while in the other the point acts as an attractor. The hyperbolic nature of the flux means that the states are extended, as expected.

There are no repulsive fixed points; hence localized states are not observed. The remaining possibility is to have critical states, as is revealed from the fact that the map contains two cycles: one of period two and the other with period six [26]. A similar stability analysis can be made for the cycles, except that the derivative is obtained by multiplying the matrices in each point of the cycle. For the period-six cycle, which corresponds to energies at the centre of the spectrum, the eigenvalues of the resulting matrix are [26]

$$r_1 = -1, \quad r_{2,3} = [(1 + 4(1 + \lambda^2)^2)^{1/2} \pm 2(1 + \lambda^2)]^2. \quad (15)$$

From this, one can show that the trace scales as $\tau_{l+6}(E) = (\tau_l(E))^\alpha$, where

$$\alpha = \ln \sigma^6 / \ln(r_2) = \ln \sigma^3 / \ln((1 + 4(1 + \lambda^2)^2)^{1/2} + 2(1 + \lambda^2)).$$

Using equation (12), the stability of the map leads to a prediction that the scaling exponent of the wavefunction is

$$\beta = \frac{\alpha}{2}. \quad (16)$$

The result for the off-diagonal problem (where $V(n) = 0$ and t_n is given by a Fibonacci sequence of t_A and t_B) is similar, but λ must be replaced by $\lambda = |y - (1/y)|/2$, where $y = t_A/t_B$. The result obtained can be compared with the scaling exponent of the state at $E = 0$. This energy is the only one for which an analytical expression is known [26]:

$$\beta = \ln y / \ln \sigma^3. \quad (17)$$

Later, a different result for the same state was published, obtained using a multifractal analysis [32]. Two limiting exponents were obtained; one is a maximum and the other a minimum [32]:

$$\beta_{\pm} = [\ln L(y^2) \pm \ln(y^2)] / \ln \sigma^3 \quad (18)$$

where $L(x)$ is defined as

$$L(x) = (1/2x)[(x+1)^2 + ((x+1)^4 + 4x^2)^{1/2}]. \quad (19)$$

In figure 2, a comparison between equations (17) and (18) and our work, given by equation (16), is shown. Notice that for $y \gg 1$, all the results coincide (for $y \ll 1$, all the methods also agree, due to the symmetrical form of λ with respect to t_A and t_B). In fact, if in equation (16) we use that $1/y$ is small compared with y when $y \gg 1$, β_- is recovered, and our method predicts exactly the same minimum scaling exponent, as expected.

For $y \rightarrow 1$, the previously published results are different ($\beta = 0$ in [26], and $\beta = 1$ in [32]). No comments were made by the respective authors about this discrepancy [26, 32], but it has been proved that for this state, the scaling exponent depends on the boundary conditions and shifting position of the chain [33]; these effects are in fact due to a general instability of singular

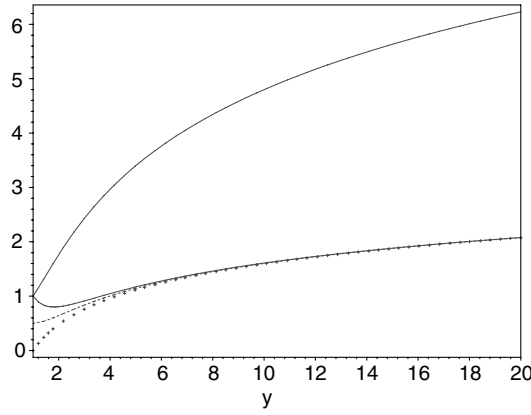


Figure 2. Comparison between the scaling exponents of the wavefunction using the method of this work (dashed curve), with the only energy for which there are analytical expressions. Crosses are obtained from Tang *et al* [26] and the two solid curves are from Fujiwara *et al* [32]. The lower curve corresponds to the minimum scaling exponent (β_-) and the other to the maximum scaling exponent (β_+).

continuous spectra [34–36] and are enhanced in this limit because it corresponds to the ‘strong quasiperiodic’ limit.

In this limit, the method presented here (equation (16)) predicts a scaling exponent $\beta = 1/2$, if one assumes that the scaling of the trace is governed by the period-six cycle. Our result is between the scaling exponents of [26] and [32], since the results from other works represent extreme cases for different classes of boundary conditions. Notice that for $y = 1$, the scaling of the trace is such that each individual band touches the neighbouring bands, and the spectrum degenerates into a single band. For $y = 1$, the six cycle degenerates into a single fixed point and extended states appear, as studied previously. However, the case $y \rightarrow 1$ but $y \neq 1$ needs more reflection. There the effects of quasiperiodicity are stronger, and a lot of problems are encountered, because the well known problem of small divisors shows up strongly in a perturbative approach [37]. In fact, when $y \rightarrow \infty$, a perturbative method can be used to obtain the spectrum [38]. The success of our initial estimation is due to the same factor; i.e., in the assumption that W_n scales as $n^{-2\beta(E,n)}$, the size of the band was estimated as a perturbation using the ‘unperturbed’ wavefunctions of each cell of the supercrystal. This approximation holds for well separated bands, but if the gaps are small compared with the bandwidths, a hybrid wavefunction can be formed. Thus, in the limit $y \rightarrow 1$ the scaling of neighbouring bands influences the scaling of the wavefunction. Since the six cycle has the lowest scaling exponent, the effect of the other bands is to increase localization, and, for example, in the Fibonacci chain, the maximum scaling exponent is given by the eigenvalues of the derivative matrix evaluated in a two cycle, which in the limit $y \rightarrow 1$ goes to 1.5.

4. Conclusions

To summarize, this work shows that the renormalization group is a very useful tool for studying localization and gives a natural classification for the types of localization, avoiding the problems concerning the thermodynamical limit that the usual LEs have. Further, the localization properties are obtained from the stability of the renormalization group, and this allows a natural classification of the types of spectrum and localization. When applied to the Fibonacci chain,

the method produces results that are comparable to those found in other works. Although the approach used only gives an approximate bound for localization, the main advantage of the method presented is that it can be applied to all of the bands. Moreover, another important issue that is clarified is that extended states are observed for $\lambda = 0$, and thus the transition from critical to extended states occurs in a discontinuous manner, as in a phase transition, when the correlation length becomes infinite. The possibility of localized states is ruled out due to the absence of repulsive fixed points.

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Queries for IOP paper

Journal: JPhysCM

Author: G G Naumis

Short title: Renormalization group stability as a diagnostic tool

Page 1

Query 1:

Author: In several places I have standardized the English somewhat. Please check that I have not inadvertently altered the meaning (and please accept my apologies if I have).

Page 2

Query 2:

Author: Amended wording 'example' OK?

Page 4

Query 3:

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Query 4:

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Page 6

Query 5:

Author: Subscript 's' Roman (for 'supercrystal' or 's band' perhaps) throughout—OK?

Page 10

Query 6:

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