Quasicrystalline and Rational Approximant Wave Patterns in Hydrodynamic and Quantum Nested Wells

A. Bazán and M. Torres
Instituto de Física Aplicada, Consejo Superior de Investigaciones Científicas, Serrano 144, 28006 Madrid, Spain

G. Chiappe, E. Louis, and J. A. Miralles
Departamento de Física Aplicada and Unidad Asociada del Consejo Superior de Investigaciones Científicas, Universidad de Alicante, San Vicente del Raspeig, Alicante 03690, Spain

J. A. Vergés
Departamento de Teoría de la Materia Condensada, Instituto de Ciencia de Materiales de Madrid, Consejo Superior de Investigaciones Científicas, Cantoblanco, Madrid 28049, Spain

Gerardo G. Naumis
Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, 01000 D.F., México

J. L. Aragón
Centro de Física Aplicada y Tecnología Avanzada, Universidad Nacional Autónoma de México, Apartado Postal 1-1010, Querétaro 76000, México
(Received 29 May 2006; published 20 September 2006)

The eigenfunctions of nested wells with an incommensurate boundary geometry, in both the hydrodynamic shallow water regime and quantum cases, are systematically and exhaustively studied in this Letter. The boundary arrangement of the nested wells consists of polygonal ones, square or hexagonal, with a concentric immersed, similar but rotated, well or plateau. A rich taxonomy of wave patterns, such as quasicrystalline states, their crystalline rational approximants, and some other exotic but well known tilings, is found in these mimicked experiments. To the best of our knowledge, these hydrodynamic rational approximants are presented here for the first time in a hydrodynamic-quantum framework. The corresponding statistical nature of the energy level spacing distribution reflects this taxonomy by changing the spectral types.

DOI: 10.1103/PhysRevLett.97.124501 PACS numbers: 47.35.−i, 47.54.−r, 71.23.Ft, 73.21.Fg

One would believe that a quasiperiodic wave pattern that has an orientational order without periodic translational symmetry [1] must be associated with an external single connected boundary. In such a case, classical analogues which model features of quantum systems, and prove nontrivial properties of these systems, have stirred interest. For example, acoustic [2] and hydrodynamic quasicrystals [3] were previously reported, and the existence of Bloch-like states has recently been proved in such systems [4]. In all of these results, there was an imposed global quasiperiodicity by either the boundary conditions or a dynamical source, which makes the appearance of such patterns not so unexpected. However, it is also possible to confine quasicrystalline hydrodynamic modes within an inner isolated region of a bigger surface, as was done in Ref. [5]. In that work, the experiment was realized under a linear regime, and, thus, the fluid "sees" a double concentric nonconnected boundary. Hence, it was essentially different from other reported nonlinear quasicrystalline Faraday wave patterns [6] where the pattern is not conditioned by the shape of the boundary due to the nonlinearity of the Faraday phenomenon. In the study of quasicrystalline structures in bounded hydrodynamic flows [3–6], the transition from order to chaos phenomena plays an important role [7].

Since the quantum analogs of confined hydrodynamic modes may be of relevance to design quantum confinements exhibiting quasiperiodic electronic states, or their rational approximants, here we systematically study the eigenfunctions corresponding to nested wells under the shallow water regime [8], i.e., in which the surface wavelength is much larger than the liquid depth, and their corresponding quantum analogues.

The nested wells of our study consist of a polygonal external boundary, square or hexagonal, with a concentric similar but incommensurately rotated well or plateau. In the case of the squares, the rotation angle between both domains is 45°, and, in the case of the hexagons, this angle is 30°. The bottom of the vessel was covered with a shallow liquid layer of depth h_1 and the inner well or plateau with a depth h_2. See Fig. 1 (bottom right).

In the hydrodynamic study, we use the equation $\partial_t^2 \eta(\mathbf{r}, t) = \nabla \cdot [g h(\mathbf{r}) \nabla \eta(\mathbf{r}, t)]$ [8], where η is the wave amplitude, g is the acceleration due to gravity, and $h(\mathbf{r})$ is the depth field, being the corresponding Helmholtz standing wave equations $(\nabla^2 + \omega^2/c_i^2)\Psi(\mathbf{r}) = 0$, $i = 1, 2$,

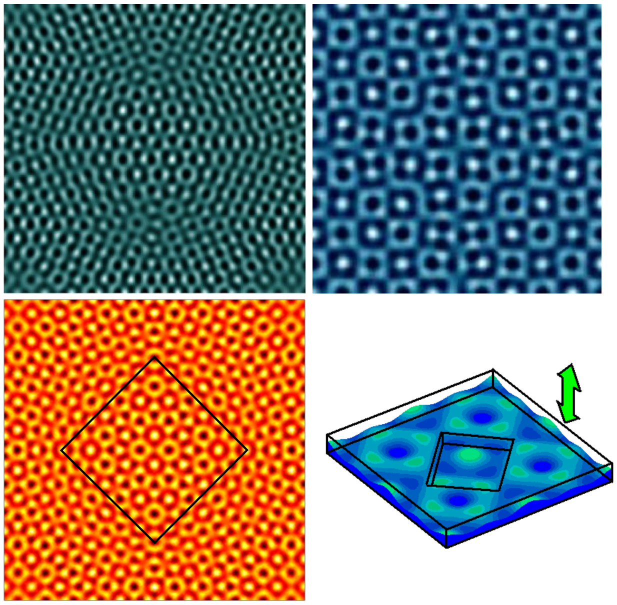

FIG. 1 (color online). Quantum octagonal quasicrystalline pattern (top left) and its 4/3 rational approximant (top right) in a double square nested well. For the quasicrystalline pattern, the well depth ratio is 1.16 and the level energy, with respect to the 50 Hz ground state, is −0.349 174. For the rational approximant, the well depth ratio is 1.1 and the level energy is −0.2246. (Bottom left) Hydrodynamic octagonal quasicrystalline pattern; the depth ratio is 0.9 and the frequency is 3.026 32 Hz. (Bottom right) Diagram of the hydrodynamic experimental vibrating setup; the edge lengths of the double square nested well are 80 and 35 cm.

where $\Psi(\mathbf{r})$ is the amplitude of the corresponding standing wave, with the above mentioned adequate regions, 1 (outer) and 2 (inner), and c_i are the phase velocities, $\omega/k_i = (gh_i)^{1/2}$, ω is the angular frequency, and k is the wave number, and $\omega = c_i k_i$ are the corresponding linear dispersion relations.

There is an analogy between the discretized Helmholtz equation and a tight-binding Hamiltonian. Discretization of the Helmholtz equation on a square lattice leads to

$$\frac{gh_{m,n}}{a^2}(\Psi_{m+1,n} + \Psi_{m-1,n} + \Psi_{m,n+1} + \Psi_{m,n-1} - 4\Psi_{m,n}) = -\omega^2\Psi_{m,n}, \tag{1}$$

where $\Psi_{m,n}$ and $h_{m,n}$ are the amplitude of the stationary wave and the liquid depth on site (m, n), respectively, and $a = L/N$, where L is the length of the external polygon and N is the discretization size (typically 201). The above equation is an eigenvalue problem formally equivalent to a quantum (tight-binding) Hamiltonian on a square lattice, represented in an orthonormal basis of atomic orbitals $|\phi_{m,n}\rangle$ ($m = 1, \ldots, N$ and $n = 1, \ldots, N$) with energy $\epsilon_{m,n} = 4gh_{m,n}/a^2$, and a hopping integral between orbitals on nearest-neighbor sites $t_{m,n;m-1,n} = t_{m,n;m+1,n} = t_{m,n;m,n-1} = t_{m,n;m,n+1} = gh_{m,n}/a^2$. The equivalence is completed if $\Psi_{m,n}$ are reinterpreted as the amplitude of the atomic orbitals in the eigenfunctions of the quantum Hamiltonian; i.e., $|\Phi\rangle = \sum_{m,n}\Psi_{m,n}|\phi_{m,n}\rangle$.

In particular, the tight-binding Hamiltonian operator that leads to Eq. (1), written in terms of the parameters and the basis set given above, is

$$\hat{H} = \sum_{m,n}\epsilon_{m,n}|\phi_{m,n}\rangle\langle\phi_{m,n}| - \sum_{\langle mn;m'n'\rangle} t_{m,n;m'n'}|\phi_{m,n}\rangle\langle\phi_{m',n'}|, \tag{2}$$

where $\langle\,\rangle$ denotes that the sum is restricted to nearest neighbors. Note that the electron mass dose not explicitly show up in this Hamiltonian. Actually, it is implicit in the hopping integral that, in turn, accounts for the electron kinetic energy. Following the stated analogy and by using linear dispersion relations, k_i vs ω, electron explicit kinetic energies would be $E_i = \hbar^2 k_i^2/2m_i = \hbar^2\omega^2 a^2/2m_i g h_i$, $i = 1, 2$, where $\hbar$ is the reduced Planck constant and m_i are electron effective masses. Hydrodynamic and quantum spectra with about 1.700 and 10.000 eigenvalues were, respectively, explored for each depth ratio h_1/h_2.

We present here quasicrystalline wave patterns and their lower rational approximants linked to the above mentioned hydrodynamic-quantum analogy problem. In Fig. 1 (top), we show a quantum octagonal quasicrystalline pattern and one of its lower rational approximants. The first pattern is obtained under quantum boundary conditions, i.e., Dirichlet conditions, and the second one, although it is also a quantum pattern, is, however, obtained under hydrodynamic boundary conditions, i.e., Neumann conditions. Figure 1 (top left) conspicuously bears resemblance to that pioneer experimental octagonal pattern published early in the very different nonlinear Faraday wave context [6]. A quasicrystalline octagonal pattern is also obtained under our present hydrodynamic framework ,and it is shown in Fig. 1 (bottom left). Quasicrystalline patterns obtained under different boundary conditions show only negligible differences. These patterns cover only the inner region of the vessel.

The quasicrystalline octagonal pattern is generated by using linear combinations of two square lattice vector bases, shifted between them by an angle of $2\tan^{-1}(2^{1/2} - 1)$. By changing $2^{1/2}$ for its respective rational approximant numbers obtained from the continuous fraction expansion $2^{1/2} = [1;\bar{2}]$, the approximant patterns are generated. There are two associated successions. The main one: $1/1, 3/2, 7/5, \ldots$ and the associated one: $2/1, 4/3, 10/7, \ldots$, $2^{1/2}$ being the geometric mean of each pair of terms. Corresponding patterns are identical but appear 45° rotated in between. These wave vectors were accurately obtained for each eigenfunction by means of its corresponding pattern Fourier analysis.

In Fig. 2, we show the hydrodynamic 3/2 rational approximant pattern and the 7/5 one with their corresponding vibrational level distributions (VLD) $D(\eta_0)$. The VLD is computationally obtained starting from the expression: $D(\eta_0) = \frac{1}{\Omega}\int_{\eta=\eta_0}(dl/|\nabla\eta(\mathbf{r})|)$, where Ω is the area of the rational approximant unit cell. By replacing the **k** space by the real space, the above mentioned VLD plays the same role in the corresponding eigenfunction as the well known density of states in the whole spectrum [9,10]. The introduction of the VLD is justified by the fact that the vibration field has only 1 degree of freedom, and it is equivalent to a scalar phonon field. So, it is possible to state an exact correspondence between a tight-binding Hamiltonian and a vibrational one [9]. VLD can also exhibit van Hove singularities which smooth out gradually as the quasicrystalline order increases [10]. The result of VLD corresponding to the octagonal quasicrystalline patterns in Fig. 1 fits the early one reported by Zaslavsky *et al.* [10], making reliable the other results for rational approximants shown here. For instance, clear van Hove singularities appear in the VLD of the $3/2$ rational approximant shown in Fig. 2 (bottom right).

In Fig. 3, we show an exotic pattern resembling the β-Mn structure (or σ phase) which has been found coexisting as a crystalline approximant with the octagonal quasicrystalline phase [11]. As Fourier analyzed in this work, this pattern grows spontaneously in this mimicked experiment with the same wave vectors of the $3/2$ rational approximant one but introducing the phase $2\arctan(1/2)$ in a couple of nonorthogonal vectors of the corresponding 4D basis. The phase shift transforms the VLD of the pattern, as shown in Fig. 3 (right), making it similar to that of the well known square lattice case and showing a clear van Hove singularity for the level zero [9,10]. So, VLD is a powerful mathematical tool to discriminate structures with similar Fourier transforms but with different internal phases.

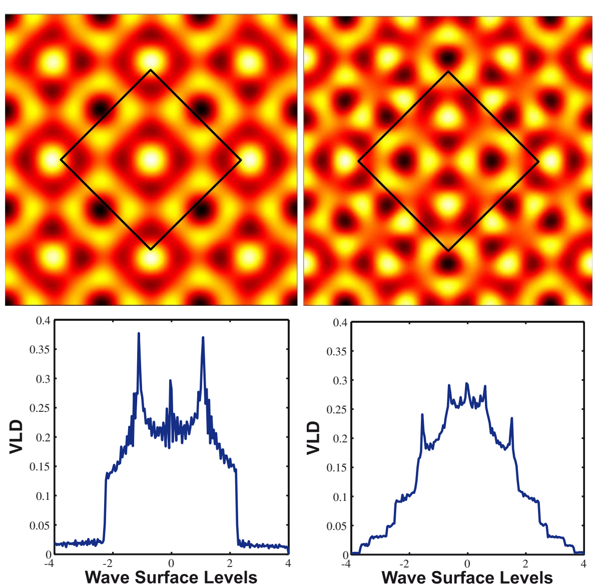

FIG. 2 (color online). Hydrodynamic rational approximants of the octagonal pattern: (top left) $3/2$ and (top right) $7/5$. In both cases, the depth ratio is 0.9 and the frequencies for each case are 0.837 670 3 Hz ($3/2$ approximant) and 0.950 524 5 Hz ($7/5$ approximant). Below each pattern, the corresponding VLD is shown.

By imposing quantum Dirichlet boundary conditions to the hydrodynamic patterns obtained, keeping the same physical conditions as those shown in Figs. 2 and 3, it is observed that the pattern phases change. So, the Neumann $3/2$ rational approximant transforms to a Dirichlet σ phase; the Neumann σ phase transforms to a Dirichlet $4/3$ rational approximant, and the Neumann $7/5$ rational approximant transforms to an experimental narrow domain containing two Dirichlet patterns: a $10/7$ rational approximant and a $7/5$ rational one with an internal phase shifting of $2\arctan(2/5)$ in its basis vector set. So, any pattern can be obtained under both, quantum or hydrodynamic, boundary conditions.

In a similar scenario, Fig. 4 shows two patterns generated in a hexagonal nested well and their corresponding level distributions. The quasicrystalline dodecagonal wave pattern can also be generated using two ternary wave vector sets shifted between them by an angle of $2\tan^{-1}(2-3^{1/2})$. Their rational approximant patterns appear when $3^{1/2}$ transforms to its corresponding rational approximant numbers. These rational numbers are obtained starting from the continuous fraction expansion $3^{1/2} = [1;\overline{1,2}]$. Figure 4 shows the 2 and $19/11$ above mentioned rational approximant wave patterns. The van Hove singularity of the VLD of the first pattern, or triangular pattern, is well known [9,10]. The spectral problem and the rich taxonomy of eigenfunctions of this double-hexagonal experiment are similar to those previously described for the double square well.

As is well known [12], there is a relationship between the geometry of the boundary, the eigenfunction nature, and the statistical properties of the spectrum. An integrable system, such as a polygonal billiard [13], has a statistical distribution of energy spectral fluctuations [$P(s)$, where s is the distance between nearest energy levels], that fits the Poisson distribution. In Fig. 5 (bottom left), we show that this is the case for the $P(s)$ of the system under the symmetric configuration of the nested wells. Because of the high symmetry of the problem, a lot of doubly degenerate states appear. If the symmetry of such a system is broken, by slightly rotating and shifting the inner well, then

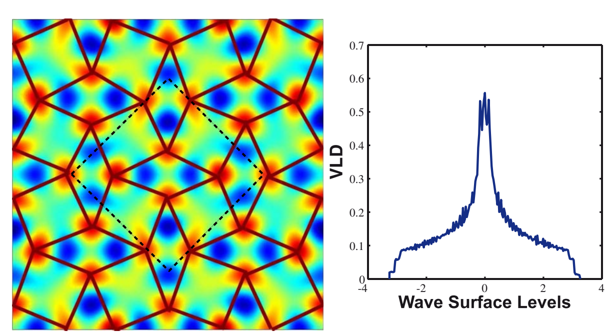

FIG. 3 (color online). Hydrodynamic pattern corresponding to the σ phase obtained with a depth ratio of 1.1 at a frequency of 0.775 317 8 Hz. Its corresponding VLD is shown at the right.

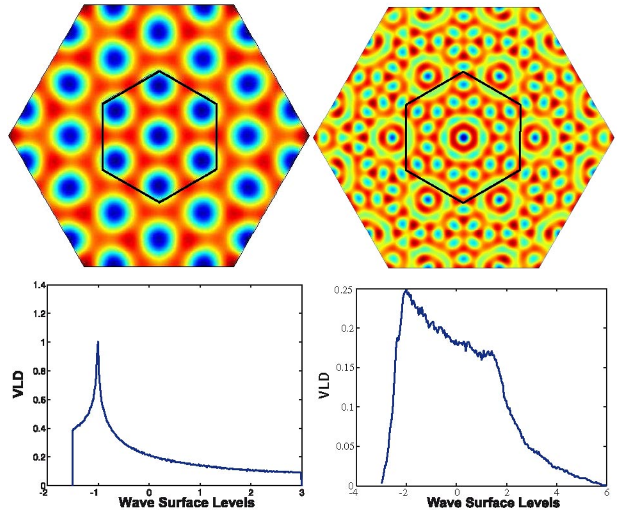

FIG. 4 (color online). Hydrodynamic triangular pattern (left) and the dodecagonal 19/11 rational approximant (right). The edge lengths of the double-hexagonal nested well are 40 and 17.5 cm. The depth ratio and frequency for each case are 1.1, 0.815 509 9 Hz (triangular) and 0.9, 2.241 697 4 Hz (19/11 approximant). Below each pattern, the corresponding VLD is shown.

all of the inherent degeneracies are removed, as was done in a similar early study [14]. In such a case, the eigenfunctions show "scarred" localized surface waves [15] as shown in Fig. 5 (top), and $P(s)$ becomes a semi-Poisson distribution, as shown in Fig. 5 (bottom right).

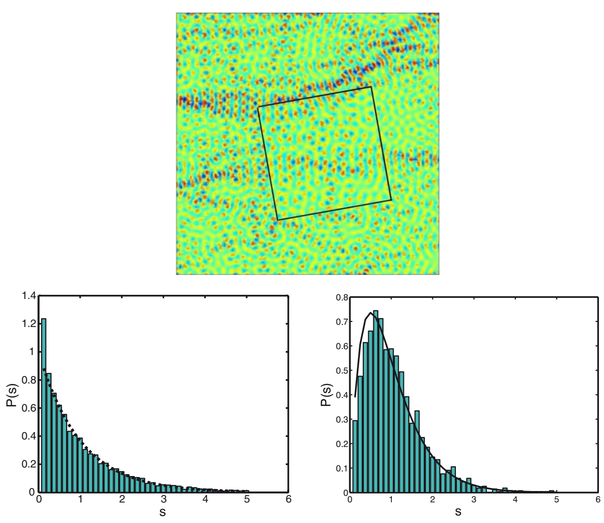

FIG. 5 (color online). (Top) Scarred chaotic wave pattern obtained when the degeneracies are removed by rotating and shifting the inner well. The depth ratio is 0.77 and the eigenfrequency is 4.149 17 Hz. (Bottom right) The corresponding semi-Poisson distribution of energy spectral fluctuations. (Bottom left) The Poisson distribution of the nearest level spacing in the whole level spectrum with degeneration due to the symmetry. The solidline represents the theoretical distribution, and the bars are experimental results.

We have shown that a geometrical arrangement of two nested wells, each of them with polygonal symmetry, can give a very rich and complex behavior even if a simple linear differential equation, valid for hydrodynamics and quantum mechanics, is used as a physical description. Quasiperiodic, approximant, and periodic wave patterns have been obtained in this work. The present approach gave a new insight on how to generate certain rational approximants in the density wave framework. The observed spectral fluctuations are consistent with the obtained wave patterns in both the symmetrical and chaotic cases.

[1] C. Janot, *Quasicrystals A Primer* (Oxford University, Oxford, 1994).
[2] S. He and J. D. Maynard, Phys. Rev. Lett. **62**, 1888 (1989); J. D. Maynard, Rev. Mod. Phys. **73**, 401 (2001).
[3] V. V. Beloshapkin, A. A. Chernikok, M. A. Natenzon, B. A. Petrovichev, R. Z. Sagdeev, and G. M. Zaslavsky, Nature (London) **337**, 133 (1989).
[4] M. Torres *et al.*, Phys. Rev. Lett. **90**, 114501 (2003).
[5] M. Torres *et al.*, Philos. Mag. **86**, 1065 (2006).
[6] B. Christiansen, P. Alstrom, and M. T. Levinsen, Phys. Rev. Lett. **68**, 2157 (1992); W. S. Edwards and S. Fauve, Phys. Rev. E **47**, R788 (1993); D. Binks and W. van de Water, Phys. Rev. Lett. **78**, 4043 (1997); R. Lifshitz and D. M. Petrich, Phys. Rev. Lett. **79**, 1261 (1997); P. Chen and J. Viñals, Phys. Rev. Lett. **79**, 2670 (1997).
[7] V. V. Beloshapkin and G. M. Zaslavskii, Phys. Lett. **97A**, 121 (1983); R. Ramshankar and J. P. Gollub, Phys. Fluids A **3**, 1344 (1991); J. P. Gollub, Proc. Natl. Acad. Sci. U.S.A. **92**, 6705 (1995).
[8] T. Chow, J. Fluid Mech. **369**, 333 (1998); M. Torres *et al.*, Phys. Rev. E **63**, 011204 (2001).
[9] E. N. Economou, *Greens Functions in Quantum Physics* (Springer-Verlag, Berlin, 1990).
[10] G. M. Zaslavsky *et al.*, *Weak Chaos and Quasi-Regular Patterns* (Cambridge University Press, Cambridge, England, 1991); A. A. Chernikov, R. Z. Sadgeev, D. A. Usikov, and G. M. Zaslavsky, Phys. Lett. A **125**, 101 (1987); A. V. Kuzmenko, I. R. Sadgeev, D. A. Usikov, and G. M. Zaslavsky, Phys. Lett. A **130**, 381 (1988); J. H. Lowenstein, Chaos **2**, 413 (1992), and references therein.
[11] N. Wang, H. Chen, and K. H. Kuo, Phys. Rev. Lett. **59**, 1010 (1987); L. Pauling, Proc. Natl. Acad. Sci. U.S.A. **85**, 2025 (1988); Z. H. Mai *et al.*, Phys. Rev. B **40**, 12 183 (1989); Z. H. Mai and L. Xu, in *Crystal-Quasicrystal Transitions*, edited by M. J. Yacamán and M. Torres (Elsevier, Amsterdam, 1993), pp. 49–81.
[12] T. A. Brody *et al.*, Rev. Mod. Phys. **53**, 385 (1981); G. M. Zaslavsky, Phys. Rep. **80**, 157 (1981).
[13] J. Wiersig, Phys. Rev. E **64**, 026212 (2001).
[14] P. J. Richens and M. V. Berry, Physica (Amsterdam) **2D**, 495 (1981).
[15] A. Kudrolli, M. C. Abraham, and J. P. Gollub, Phys. Rev. E **63**, 026208 (2001), and references therein.