# Point charge in a three-dielectric medium with planar interfaces

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A simple method of obtaining integral representations of the electrostatic potential, the induced surface charge density, and the image potential for a point charge in a three-dielectric medium with infinite planar interfaces is presented. The total induced charge at the interfaces is readily evaluated. Numerical results for the image potential in several illustrative cases are also shown. The case of a point charge between two grounded conducting plates becomes a special limiting case of the present problem.

## I. INTRODUCTION

The electrostatic potential of a point charge between two grounded conducting plates is well known[1] since it can be immediately expressed in terms of infinite series using the method of images. An alternative method is to treat the problem as a boundary-value problem and to express the potential as an eigenfunction expansion.[2,3] Nevertheless interest in this problem continues since these infinite series are not very useful in practical calculations and alternative forms of the solution are desirable. Pumplin[4] and Glasser,[5] for example, obtain a simple integral representation of the potential through a complicated series of integral transformations. It is also known[6] that the calculation of the total charge induced at the plates through the method of images leads to convergence problems which have to be dealt with subtle delicacy.[3,7,8]

In this paper, we demonstrate that the method of images can be used to obtain an integral representation of the potential through an extremely simple procedure even for the more general problem of a point charge in a three-dielectric medium with planar interfaces. The procedure for evaluating the potential due to the infinite array of images is derived from the observation that a two-dimensional (parallel to the interfaces) Fourier transform of their Coulomb potentials leads to a geometric series which can be readily summed. Transforming back to $\bar{r}$ space immediately yields the potential in integral form. The calculation of other electrostatic quantities is straightforward and the resulting expressions have a simple form.

In Sec. II we calculate explicitly the electrostatic potential of the system in all regions of space for an arbitrary position of the point charge. The potential of a point charge between two grounded conducting plates separated by vacuum becomes a special limiting case of our general expression for the potential which, in this limit, reduces to Glasser's form.[5] In Sec. III we use the formula for the potential derived in Sec. II in order to obtain an integral expression for the surface charge density induced at the interfaces for an arbitrary position of the point charge. The total charge induced at the interfaces is also readily evaluated without any convergence problems or the use of any special artifice even in the limiting case when one or two of the media are conductors. Section IV is devoted to the calculation of an integral expression for the image potential which in the case of two grounded conductors separated by vacuum can be integrated in closed form. For other illustrative cases we perform the integral numerically. In Sec. V we present a synopsis of our results.

## II. THE ELECTROSTATIC POTENTIAL

We consider a point charge $Q$ imbedded in an inhomogeneous system with planar interfaces perpendicular to the $z$ axis and characterized by the sequence of dielectric constants:

$$\begin{aligned} &\epsilon_1, \quad z < -a;\\ &\epsilon_2, \quad -a < z < a;\\ &\epsilon_3, \quad z > a. \end{aligned} \tag{1}$$

Without loss of generality we locate the point charge on the $Z$ axis at $z = z'$. The geometry of the problem is shown in Fig. 1 where $R$, $z$, and $\phi$ are cylindrical coordinates. Since our problem has azimuthal symmetry the potential in cylindrical coordinates does not depend on $\phi$. Therefore we introduce the notation $v(R,z;z')$ as the potential at $(R,z)$ due to a point charge at $z = z'$. For simplicity and continuity in the presentation we describe our procedure first by considering the case $-a < z' < a$ and by calculating the potential only in the region $-a < z < a$. The solutions for the other regions of space and for the other possible point charge positions are obtained through similar manipulations, and we only quote the results in a separate table.

Using the method of images[9,10] a straightforward calculation (for $-a < z' < a$ and $-a < z < a$) yields

$$\begin{aligned} v(R,z;z') = \frac{Q}{\epsilon_2}\Bigg[&\frac{1}{[R^2+(z-z')^2]^{1/2}} - \sum_{n=0}^{\infty}(L_{12}L_{32})^n\left(\frac{L_{12}}{[R^2+(z+z'+|4n+2|a)^2]^{1/2}}\right.\\ &\left.+\frac{L_{32}}{[R^2+(z+z'-|4n+2|a)^2]^{1/2}}\right) + \sum_{n=1}^{\infty}(L_{12}L_{32})^n\left(\frac{1}{[R^2+(z-z'-4na)^2]^{1/2}}\right.\\ &\left.+\frac{1}{[R^2+(z-z'+4na)^2]^{1/2}}\right)\Bigg], \end{aligned} \tag{2a}$$

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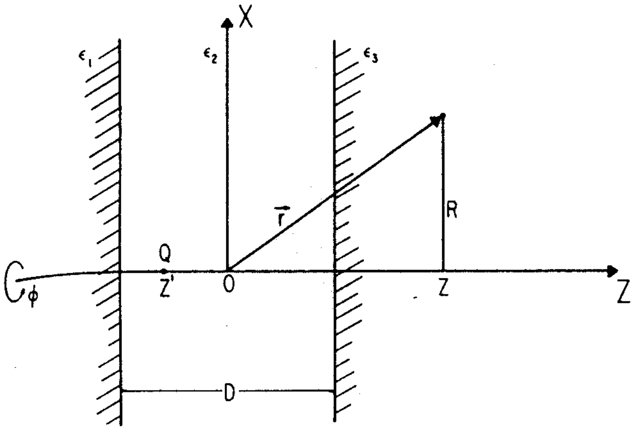


Fig. 1. Cross section of the system along a plane of symmetry.

where

$$L_{i2} \equiv (\epsilon_i - \epsilon_2)/(\epsilon_i + \epsilon_2); \qquad i = 1,3. \tag{2b}$$

We define the two-dimensional Fourier transform $F(\kappa,z)$ of a function $f(\mathbf{R},z)$ as

$$F(\kappa,z) = \int d^2R\, f(\mathbf{R},z) e^{-i\kappa\cdot\mathbf{R}}, \tag{3}$$

where $\kappa$ and $\mathbf{R} \equiv (R,\phi)$ are two-dimensional vectors parallel to the interfaces. The inverse Fourier transform is then

$$f(\mathbf{R},z) = \int \frac{d^2\kappa}{(2\pi)^2} F(\kappa,z) e^{i\kappa\cdot\mathbf{R}}. \tag{4}$$

Therefore the Fourier transform of the Coulomb potential of a point charge $Q$ at $z = z'$

$$f(\mathbf{R},z;z') = \frac{Q}{|R^2 + (z-z')^2|^{1/2}} \tag{5}$$

is

$$F(\kappa,z;z') = Q(2\pi/\kappa) e^{-\kappa|z-z'|}. \tag{6}$$

Taking now the Fourier transform on both sides of Eq. (2) yields

$$V(\kappa,z;z') = \frac{Q}{\epsilon_2}\frac{2\pi}{\kappa}\Big\{e^{-\kappa|z-z'|} - \Big(L_{12}e^{-\kappa|z+z'+2a|} + L_{32}e^{-\kappa|z+z'-2a|}\sum_{n=0}^{\infty}(L_{12}L_{32}e^{-4\kappa a})^n + (e^{-\kappa(z-z')} + e^{\kappa(z-z')})\sum_{n=1}^{\infty}(L_{12}L_{32}e^{-4\kappa a})^n\Big)\Big\}. \tag{7}$$

Since a geometric series is readily summed

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \qquad -1 < r < 1; \tag{8}$$

Eq. (7) becomes

$$V(\kappa,z;z') = (Q/\epsilon_2)(2\pi/\kappa)\{e^{-\kappa|z-z'|} - \Delta(\kappa,D)[L_{12}e^{-\kappa|z+z'+D|} + L_{32}e^{-\kappa|z+z'-D|} - L_{12}L_{32}e^{-2\kappa D}(e^{-\kappa|z-z'|} + e^{\kappa(z-z')})]\}, \tag{9a}$$

where $D \equiv 2a$ is the separation between media 1 and 3, and

$$\Delta(\kappa,D) \equiv [1 - L_{12}L_{32}e^{-2\kappa D}]^{-1}. \tag{9b}$$

In Eq. (9a) the first term on the right-hand side is the potential produced by the point charge and the next terms correspond to the potential produced by the charge induced at the interfaces. For other regions of space and for other possible positions of the point charge the Fourier transform $V(\kappa,z;z')$ of the electrostatic potential is calculated through the same procedure. The results for all different cases are shown in Table I.

Table I. This table shows, in all regions of space, the two-dimensional Fourier transforms $V(R,z;z')$ of the electric potential produced by a point charge $Q$, located on the $Z$ axis at $z = z'$, in a three-dielectric medium with planar interfaces. The medium is characterized by the dielectric constants $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$ for the regions $z < -z$, $-a < z < a$, and $z > a$, respectively, and $\kappa$ is a two-dimensional vector parallel to the interfaces.

| Region of space | $V(\kappa,z;z')$ |
|---|---|
| | Case I: $-a < z' < a$ |
| $z > a$ | $\left(\frac{2Q}{\epsilon_2+\epsilon_3}\right)\frac{2\pi}{\kappa}\Delta(\kappa,D)[e^{-\kappa\lvert z-z'\rvert} - L_{12}e^{-\kappa(D+\lvert z+z'\rvert)}]$ |
| $-a < z < a$ | $\frac{Q}{\epsilon_2}\frac{2\pi}{\kappa}\{e^{-\kappa\lvert z-z'\rvert} - \Delta(\kappa,D)[L_{12}e^{-\kappa\lvert z+z'+D\rvert} + L_{32}e^{-\kappa\lvert z+z'-D\rvert} - L_{12}L_{32}e^{-2\kappa D}(e^{-\kappa\lvert z-z'\rvert} + e^{\kappa\lvert z-z'\rvert})]\}$ |
| $z < -a$ | $\left(\frac{2Q}{\epsilon_2+\epsilon_1}\right)\frac{2\pi}{\kappa}\Delta(\kappa,D)[e^{-\kappa\lvert z+z'\rvert} - L_{32}e^{-\kappa\lvert D-z+z'\rvert}]$ |
| | Case II: $z' > a$ |
| $z > a$ | $\frac{Q}{\epsilon_3}\frac{2\pi}{\kappa}\left[e^{-\kappa\lvert z-z'\rvert} + L_{32}e^{-\kappa\lvert z+z'-D\rvert} - \left(\frac{2\epsilon_3}{\epsilon_3+\epsilon_2}\right)\left(\frac{2\epsilon_2}{\epsilon_2+\epsilon_3}\right)L_{12}\Delta(\kappa,D)e^{-\kappa(D+\lvert z\rvert+\lvert z'\rvert)}\right]$ |
| $-a < z < a$ | $\left(\frac{2Q}{\epsilon_3+\epsilon_2}\right)\frac{2\pi}{\kappa}\Delta(\kappa,D)[e^{-\kappa\lvert z-z'\rvert} - L_{12}e^{-\kappa(D+\lvert z+z'\rvert)}]$ |
| $z < -a$ | $Q\left(\frac{2}{\epsilon_2+\epsilon_1}\right)\left(\frac{2\epsilon_2}{\epsilon_2+\epsilon_3}\right)[e^{-\kappa\lvert z-z'\rvert} - e^{-\kappa(\lvert z\rvert+\lvert z'\rvert)} + \Delta(\kappa,D)e^{-\kappa(\lvert z\rvert+\lvert z'\rvert)}]$ |
| | Case III: $z' < -z$ |
| | Case III is obtained directly from case II by replacing $\epsilon_1 \leftrightarrow \epsilon_3$ and $z \to -z$. |

In order to obtain an integral expression for the potential we transform $V(\kappa,z;z')$ back to $R$ space using Eq. (4). Since the problem has cylindrical symmetry, we can write

$$v(\mathbf{R},z;z') = \int_0^\infty \frac{\kappa\, d\kappa}{(2\pi)^2} V(\kappa,z;z') \times \int_0^{2\pi} d\phi\, e^{i\kappa R \cos\phi}, \quad (10)$$

where the axis from which $\phi$ is defined is irrelevant. But the $\phi$ integral is an integral representation of the zero-order Bessel function[11]

$$J_0(X) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{iX\cos\phi} \quad (11)$$

thus we can write the electrostatic potential

$$v(\mathbf{R},z;z') = \frac{1}{2\pi}\int_0^\infty \kappa\, d\kappa J_0(\kappa R) V(\kappa,z;z') \quad (12)$$

as a single definite integral where $V(\kappa,z;z')$ is given in Table I.

Since a perfect conductor is characterizd by an infinite static dielectric constant the potential of a point charge between two grounded conducting media separated by vacuum is obtained directly from Eq. (9) by taking the limit $\epsilon_2 \to 1$ and $\epsilon_1 = \epsilon_3 \to \infty$ (or equivalently $L_{12} = L_{32} \to 1$). It is a property of the image method[9] that in this limit (metallic limit) both conducting media become grounded (thus in short-circuit) since the images are then constructed by keeping both interfaces at zero potential. For any other finite values of $\epsilon_2$ and $\epsilon_3$ the two media are isolated from each other.

Taking the metallic limit in Eq. (9a) we obtain

$$V(\kappa,z;z') = Q\,(2\pi/\kappa)\left[e^{-\kappa|z-z'|} - \left(\frac{\cos\kappa(z+z') - e^{-\kappa D}\cosh\kappa|z-z'|}{\sinh\kappa D}\right)\right], \quad (13)$$

where the first term on the right-hand side shows explicitly the contribution to the potential due to the point charge. It can also be seen from Table I (case I) that in this limit $V \to 0$ for $z > a$ and $z < -a$ corresponding to the case of two grounded plates as stated above. Equation (13) can also be written (for $z > z'$) as

$$V(\kappa,z;z') = Q\frac{4\pi}{\kappa} \times \left(\frac{\sinh\kappa(D/2+z')\sinh\kappa(D/2-z)}{\sinh\kappa D}\right) \quad (14)$$

which reduces to Glasser's form[5] when substituted in Eq. (12).

It is also important to realize that our general solution for the potential $v(R,z;z')$ can be used to solve an even more general problem. The function $v(R,z;z')$ is, by definition[9,12] the two-dimensional Green's function of Poisson's equation in the geometry of the present problem with the boundary condition of vanishing potential at infinity. It is called two-dimensional because it does not depend on the angular coordinates. Therefore the potential $v_\lambda(R,z)$ of an arbitrary linear charge density $\lambda(z)$ of finite extent along the $Z$ axis in the present geometry is given by superposition through the integral

$$v_\lambda(R,z) = \int dz'\, v(R,z;z')\,\lambda(z'), \quad (15)$$

where the limits of integration are determined by the extent of the linear charge distribution which has to be finite in order to fulfill the boundary condition at infinity.

## III. SURFACE CHARGE DENSITY

In this section we calculate the surface charge density $\sigma(R,\pm a;z')$ induced at the interfaces $\pm a$ for $-a < z' < a$. The calculation of $\sigma(R,\pm a;z')$ for the other possible positions of the point charge is done following the same procedure and we only quote the results in Table II. We also calculate the total charge induced at the interfaces for an arbitrary position of the point charge.

The induced surface charge density is given[13] by the discontinuity of the normal component of the electric field at the interfaces which in our case can be written

$$\sigma(R,\pm a;z') = (1/4\pi)[E_z(R,\pm a_+;z') - E_z(R,\pm a_-;z')]. \quad (16a)$$

where

$$a_\pm \equiv \lim_{\delta\to 0}(a \pm \delta). \quad (16b)$$

Table II. This table shows the two-dimensional Fourier transform of the surface charge density $\Sigma(\kappa,\pm a;z')$ induced at the interfaces at $z = \pm a$ by a point charge $Q$, located on the $Z$ axis at $z = z'$, in a three-dielectric medium with planar interfaces. The medium is characterized by the dielectric constants $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$ for the regions $z < -a$, $-a < z < a$, and $z > a$, respectively, and $\kappa$ is a two-dimensional vector parallel to the interfaces.

Case I: $-a < z' < a$

$$\Sigma(\kappa,a;z') = -Q\frac{L_{32}}{\epsilon_2}\Delta(\kappa,D)\,[e^{-\kappa|z'-a|} - L_{12}e^{-\kappa(D+|z'+a|)}]$$

$$\Sigma(\kappa,-a;z') = -Q\frac{L_{12}}{\epsilon_2}\Delta(\kappa,D)[e^{-\kappa|z'+a|} - L_{32}e^{-\kappa(D+|z'-a|)}]$$

Case II: $z' > a$

$$\Sigma(\kappa,a;z') = Q\frac{L_{32}}{\epsilon_3}\left[1 + \left(\frac{2\epsilon_3}{\epsilon_3+\epsilon_2}\right)\Delta(\kappa,D)L_{12}e^{-2\kappa D}\right]e^{-\kappa(|z'|-D/2)}$$

$$\Sigma(\kappa,-a;z') = -Q\frac{2}{\epsilon_3+\epsilon_2}\Delta(\kappa,D)L_{12}e^{-\kappa(|z'|+D/2)}$$

Case III: $z' < -a$

Case III is obtained directly from case II by replacing $\epsilon_1 \leftrightarrow \epsilon_3$ and $a \to -a$.

Here and in the following expressions the upper sign of $a$ refers to the interface at $z = +a$ and the lower sign to the interface at $z = -a$.

Since $\mathbf{E} = -\nabla v$, the Fourier component $\xi(\kappa,z;z')$ of $\mathbf{E}(\mathbf{R},z;z')$ is given by

$$\xi(\kappa,z;z') = i\kappa V(\kappa,z;z') - \hat{k}\frac{\partial}{\partial z}V(\kappa,z;z'), \qquad (17)$$

where $\hat{k}$ is the unit vector along the $Z$ axis. Thus the Fourier transform $\Sigma(\kappa,\pm a;z')$ of the induced surface charge density $\sigma(\mathbf{R},\pm a;z')$ becomes

$$\Sigma(\kappa,\pm a;z') = \frac{1}{4\pi}\left[\frac{\partial}{\partial z}V(\kappa,z;z')\Big|_{z\to +a-} - \frac{\partial}{\partial z}V(\kappa,z;z')\Big|_{z\to\pm a+}\right]. \qquad (18)$$

In case I ($-a < z' < z$) substitution of the relevant expressions of Table I into Eq. (18) yield

$$\Sigma(\kappa,z;z') = -Q(L_{32}/\epsilon_2)\Delta(\kappa,D)[e^{-\kappa|z'-a|} - L_{12}e^{-\kappa(D+|z'+a|)}] \qquad (19)$$

and the charge density $\Sigma(\kappa,-a;z')$ induced at the interface $z = -a$ is obtained from Eq. (19) by exchanging $\epsilon_1 \leftrightarrow \epsilon_3$ and changing $a \to -a$. All the other possible cases are shown in Table II.

The induced surface charge density $\sigma(\mathbf{R},\pm a;z')$ is now obtained by transforming $\Sigma(\kappa,\pm a;z')$ back to $R$ space through Eq. (4) yielding

$$\sigma(\mathbf{R},\pm a;z') = \frac{1}{2\pi}\int_0^\infty \kappa\, d\kappa\, J_0(\kappa R)\Sigma(\kappa,\pm a;z') \qquad (20)$$

in an integral form. $\Sigma(\kappa,\pm a;z')$ is given in Table II.

The total charge $q(\pm a;z')$ induced at each interface for an arbitrary position $z'$ of the point charge is obtained,

$$q(\pm a;z') = \int d^2R\, \sigma(\mathbf{R},\pm a;z') \qquad (21)$$

by integrating the induced surface charge density over the whole interface area.

Nevertheless $q(\pm a;z')$ can be readily calculated since, by comparing Eqs. (3) and (21),

$$q(\pm a;z') = \Sigma(\kappa = 0,\pm a;z'). \qquad (22)$$

Thus $q(\pm a;z')$ is simply the $\kappa = 0$ component of the induced charge density $\Sigma(\kappa,z;z')$.

Therefore in case I ($-a < z' < a$) the total charge induced at the interface is obtained from Eqs. (22) and (19) and it is given by

$$q(a;z') = -(Q/\epsilon_2)(\epsilon_3 - \epsilon_2)/(\epsilon_3 + \epsilon_1) \qquad (23a)$$

and similarly

$$q(-a;z') = -(Q/\epsilon_2)(\epsilon_1 - \epsilon_2)/(\epsilon_1 + \epsilon_3). \qquad (23b)$$

We can see that the total charge induced at the interfaces $q(\pm a;z')$ does not depend on the position $z'$ of the point charge.

The total charge $q(z')$ induced at both interfaces is simply

$$q(z') = q(-a;z') + q(a;z'); \qquad (24)$$

the combination of Eqs. (23) and (24) yields

$$q(z') = -(Q/\epsilon_2)[1 - 2\epsilon_2/(\epsilon_1 + \epsilon_3)]. \qquad (25)$$

On the other hand the total *volume* polarization charge $q_v$ induced "around" the point charge is given by[9]

$$q_v = (Q/\epsilon_2)(1 - \epsilon_2). \qquad (26)$$

Therefore the total charge $q_T = q(z') + q_v$ induced in the system is, in this case,

$$q_T = -Q[1 - 2/(\epsilon_1 + \epsilon_3)], \qquad (27)$$

different from zero. This is due to the fact that $q_T$ does not correspond to the total polarization charge induced in a dielectric system of *finite* size in which the total polarization charge is always zero. Since the width of two of the dielectric media is infinite one is not considering the contribution of the polarization charge in the outer boundaries "at infinity." This is not the case, for example, of a point charge imbedded in a dielectric slab surrounded by vacuum. In this case ($\epsilon_1 = \epsilon_3 = 1$) the total charge induced in the system is the total polarization charge and from Eq. (27) one obtains by setting $\epsilon_1 = \epsilon_3 = 1$ that $q_T = 0$ as expected.

In the limiting case of a point charge in vacuum between two grounded conducting plates ($L_{12} = L_{32} = 1$; $\epsilon_2 = 1$) Eq. (19) becomes

$$\Sigma(\kappa,\pm a;z') = -Q\frac{e^{-\kappa|z'+a|} - e^{-\kappa(D+|z'\pm a|)}}{1 - e^{-2\kappa D}}. \qquad (28)$$

According to Eq. (22) the total charge $q(\pm a;z')$ induced at each plate is the $\kappa \to 0$ limit of Eq. (28). Using L'Hospital's rule this yields the well-known result[13]

$$q(\pm a;z') = -Q[(D/2 + z')/D] \qquad (29)$$

and

$$q(z') = -Q \qquad (30)$$

in which the total charge $q(\pm a;z')$ induced at each plate depends on the position $z'$ of the point charge and adds up to $-Q$.

The calculation of $q(\pm a;z')$ through Eq. (22) has a simple physical interpretation. The Fourier transform of the electric field $\xi(\kappa,z;z')$ of a point charge $Q$ located at $z = z'$ is obtaining by combining Eqs. (6) and (17) to yield

$$\xi(\kappa,z;z') = 2\pi Q e^{-\kappa|z-z'|}[i\hat{\kappa} + \hat{k}\,\mathrm{sgn}(z - z')], \qquad (31)$$

where $\hat{\kappa}$ and $\hat{k}$ are unit vectors in the direction of $\kappa$ and the $Z$ axis, respectively. The $\kappa = 0$ component of $\xi(\kappa,z;z')$ is simply

$$\xi(\kappa = 0,z;z') = 2\pi Q\hat{k}\,\mathrm{sgn}(z - z'), \qquad (32)$$

the electric field $\mathbf{E}$ (in $R$ space) produced by an infinite plate at $z = z'$ with charge density $Q$. Thus, in the present problem, $\Sigma(\kappa = 0,\pm a;z')$ is the surface charge density induced by this plate at each interface. In conclusion, the problem of obtaining the total charge induced by a point charge at each interface of a three-dielectric medium with planar interfaces is equivalent, according to Eq. (22), to the calculation of the surface charge density induced by an infinite plate. Through the use of symmetry arguments Purcell[14] already has utilized this procedure in the calculation of the total charge induced by a point charge at the grounded plates of a capacitor. Here we show explicitly the validity of his procedure within the context of a well-defined method for the complete solution of an even more general problem.

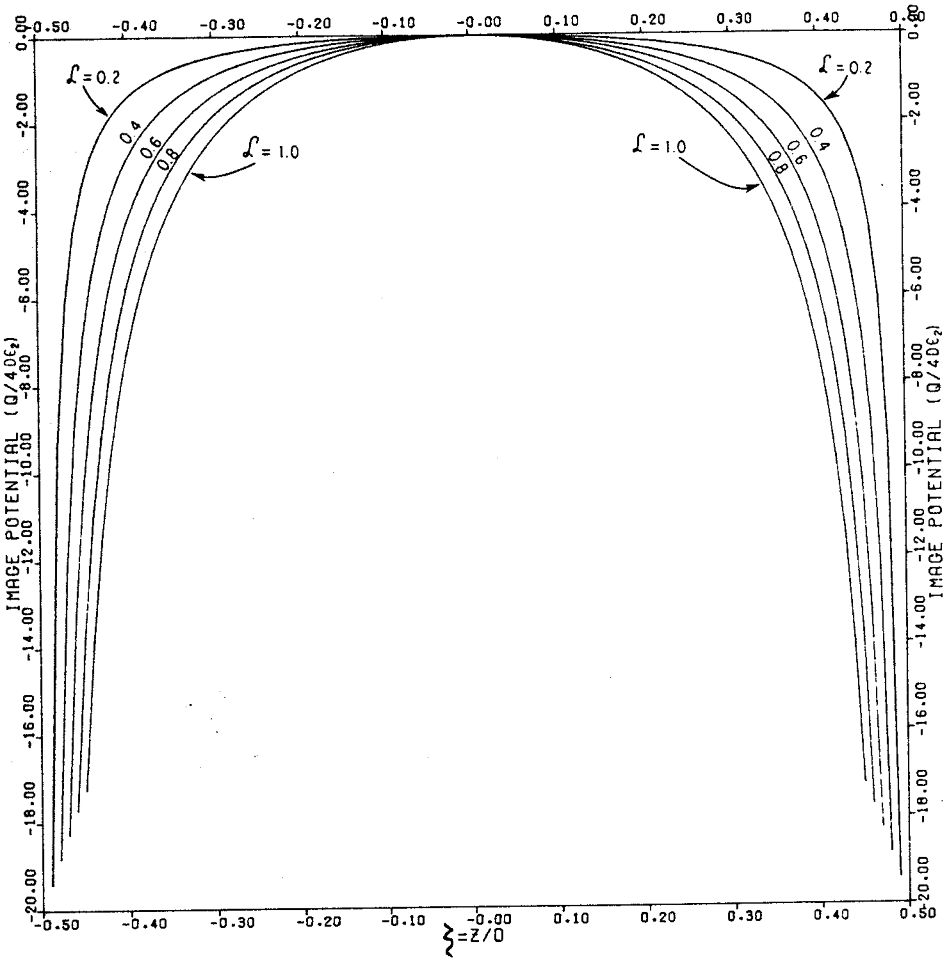


Fig. 2. Image potential in units of $Q/4D\epsilon_2$ for a point charge imbedded in a dielectric medium located between two identical semi-infinite media as a function of $\zeta\,(= z/D)$ and for different values of $\mathcal{L}$.

In case II ($z' > a$) combination of Eq. (22) with the relevant expressions of Table II yield

$$q(a,z') = Q\epsilon_1(\epsilon_3 - \epsilon_2)/\epsilon_3\epsilon_2(\epsilon_1 + \epsilon_3) \quad (33a)$$

and

$$q(-a;z') = -Q(\epsilon_1 - \epsilon_2)/\epsilon_2(\epsilon_1 + \epsilon_3) \quad (33b)$$

and the total charge induced at both interfaces becomes

$$q(z') = Q(\epsilon_3 - \epsilon_1)/(\epsilon_3 + \epsilon_1). \quad (34)$$

In the case of a point charge in front of a dielectric slab surrounded by vacuum ($\epsilon_1 = \epsilon_3 = 1$) Eq. (33) reduces to

$$q(\pm a;z') = \pm Q(1 - \epsilon_2)/2\epsilon_2, \quad (35)$$

the total charge induced at one interface is equal in magnitude but opposite in sign to the one induced at the other interface. Consequently the total induced charge at both interfaces, which in this case coincides with the total surface polarization charge, cancels. For a point charge in front of a conducting slab ($\epsilon_2 \to \infty$) Eq. (35) reduces to $q(\pm a;z') = \pm Q/2$.

In case III ($z' < a$) $q(\pm a;z')$ and $q(z')$ are obtained from Eqs. (33) and (34) by exchanging, $\epsilon_1 \leftrightarrow \epsilon_3$.

## IV. IMAGE POTENTIAL

In this section we make use of the expressions derived already in Sec. II in order to calculate the image potential of a point charge at an arbitrary position of the charge in the three-dielectric medium.

First we discuss in detail the case when the point charge is located at $-a < z' < a$ and only quote the results for $|z'| > |a|$.

By definition, the image potential $v_{\rm im}$ at $z = z'$ can be written as

$$v_{\rm im}(z') = -\int_{z0}^{z'} \mathbf{E}'\cdot d\mathbf{l}, \quad (36)$$

where $z_0$ is the origin of the potential, $\mathbf{E}'$ is the electric field at the point charge position ($\mathbf{R} = 0$, $z = z'$) due to the induced charge density at the interfaces, and the $Z$ axis is chosen as integration path.

Due to the azimuthal symmetry of the system the field $\mathbf{E}'(z')$ has only the $z$ component which can be written

$$E'_z(z') = \lim_{\substack{R\to 0\\ z\to z'}} \left(-\frac{\partial}{\partial z} v_{\rm ind}(\mathbf{R},z;z')\right), \quad (37)$$

where $v_{\rm ind}(\mathbf{R},z;z')$ is the potential produced by the induced surface charge at the interfaces in the region of interest.

$v_{\rm ind}$ is obtained from the total potential $v(R,z;z')$ of the system by substracting the potential $v_{\rm self}$ produced by the point charge itself, thus

$$v_{\rm ind}(R,z;z') = v(R,z;z') - v_{\rm self}. \quad (38)$$

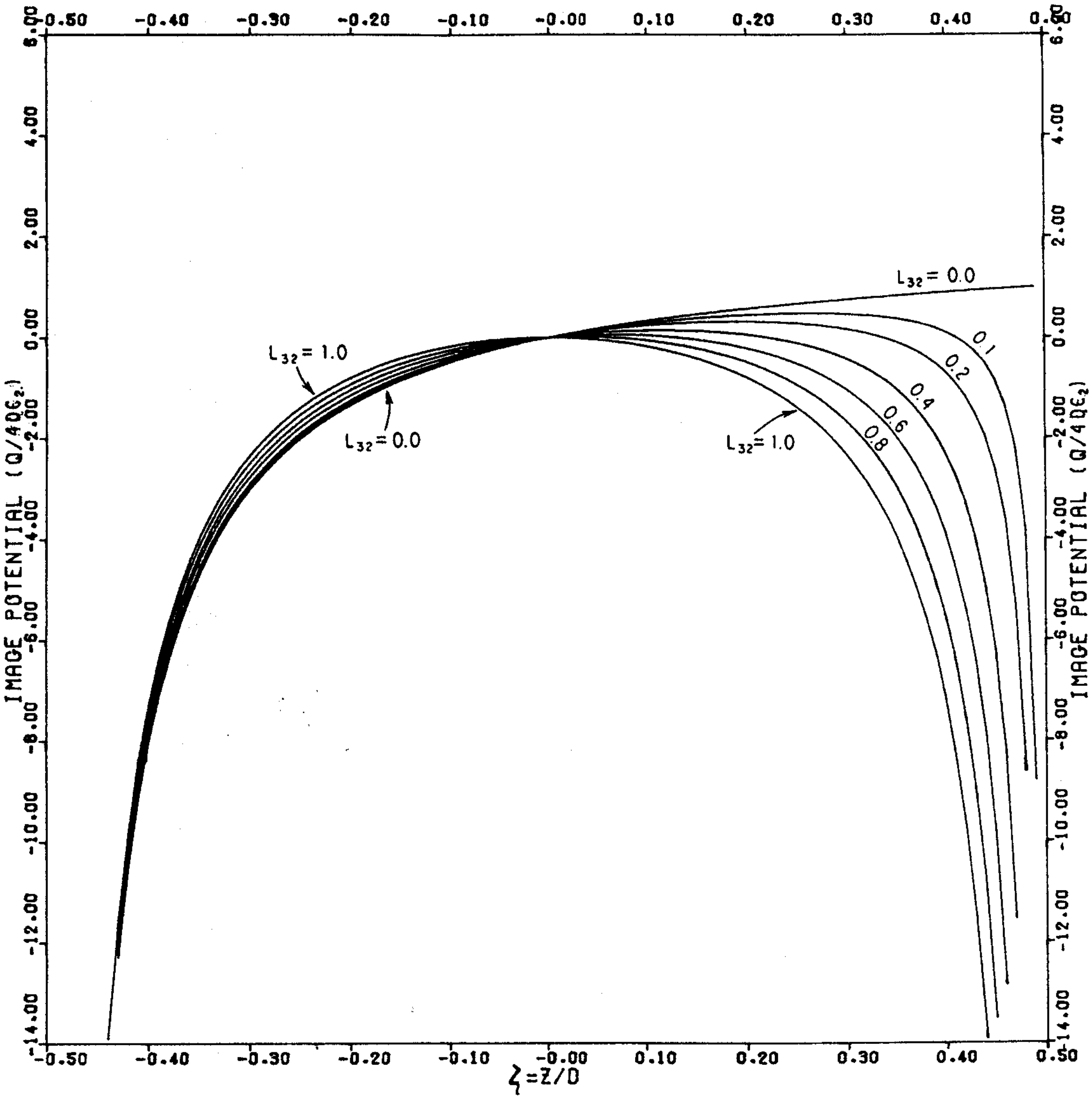


Fig. 3. Image potential in units of $Q/4D\epsilon_2$ for a point charge imbedded in a dielectric medium ($\epsilon_2$) located between a metal and a dielectric as a function of $\zeta\ (= z/D)$ and for different values of $L_{23}$.

In *case I* ($-a < z' < a$), $v_{\rm ind}$ is obtained from Eqs. (9a) and (12). Since in Eq. (9a) the first term on the right-hand side represents the contribution to the potential from the external point charge, $v_{\rm ind}$ is readily written

$$v_{\rm ind}(R,z;z') = -\frac{Q}{\epsilon_2}\int_0^\infty d\kappa\, J_0(\kappa R)\Delta(\kappa,D) \times [L_{12}e^{-\kappa|z+z'+D|} + L_{32}e^{-\kappa|z+z'-D|} - L_{12}L_{32}e^{-2\kappa D}(e^{-\kappa(z-z')} + e^{\kappa(z-z')})]. \quad (39)$$

Using Eqs. (37) and (39) we can write

$$E_z'(z') = -\frac{Q}{\epsilon_2}\int_0^\infty \kappa\, d\kappa\, \Delta(\kappa,D)e^{-\kappa D} \times (L_{12}e^{-2\kappa z'} - L_{32}e^{2\kappa z'}). \quad (40)$$

Substituting Eq. (40) into Eq. (36), choosing the zero of the potential at the origin $z_0 = 0$, changing the order of integration, and integrating over $z'$ from 0 to $z'$, we obtain

$$v_{\rm im}(z') = -\frac{Q}{2\epsilon_2}\int_0^\infty d\kappa\, \Delta(\kappa,D)e^{-\kappa D}[L_{12}e^{-2\kappa z'} + L_{32}e^{2\kappa z'} - (L_{12} + L_{32})]. \quad (41)$$

Insertion of Eq. (9b) into Eq. (41) and the following change of variables,

$$\zeta \equiv z'/D \quad \text{and} \quad \chi = 2\kappa D, \quad (42)$$

leads to the following expression:

$$v_{\rm im}(\zeta) = \frac{Q/\epsilon_2}{2D}\int_0^\infty d\chi \left[\frac{L_{12}e^{-\zeta\chi/2} - L_{32}e^{\zeta\chi/2}}{e^{\chi/2} - L_{12}L_{32}e^{-\chi/2}}\right]\sinh\frac{\zeta\chi}{2}; \quad -\tfrac{1}{2} < \zeta < \tfrac{1}{2} \quad (43)$$

which can be integrated by numerical methods.

At the metallic limit ($\epsilon_2 = L_{12} = L_{32} = 1$, Eq. (43) takes a simpler form

$$v_{\rm im}(\zeta) = \frac{Q}{2D}\int_0^\infty dy\, (e^{-\zeta y} - e^{\zeta y})\left(\frac{\sinh \zeta y}{\sinh y}\right), \quad (44)$$

where the variable of integration has been changed to $y = \chi/2$. But this integral can be expressed[15] in closed form

$$v_{\rm im}(\zeta) = (Q/4D)[\psi(\tfrac{1}{2} - \zeta) + \psi(\tfrac{1}{2} + \zeta) - 2\psi(\tfrac{1}{2})] \quad (45)$$

in terms of the $\psi$ (digamma) function.[15]

Numerical integration of Eq. (43) is performed in order

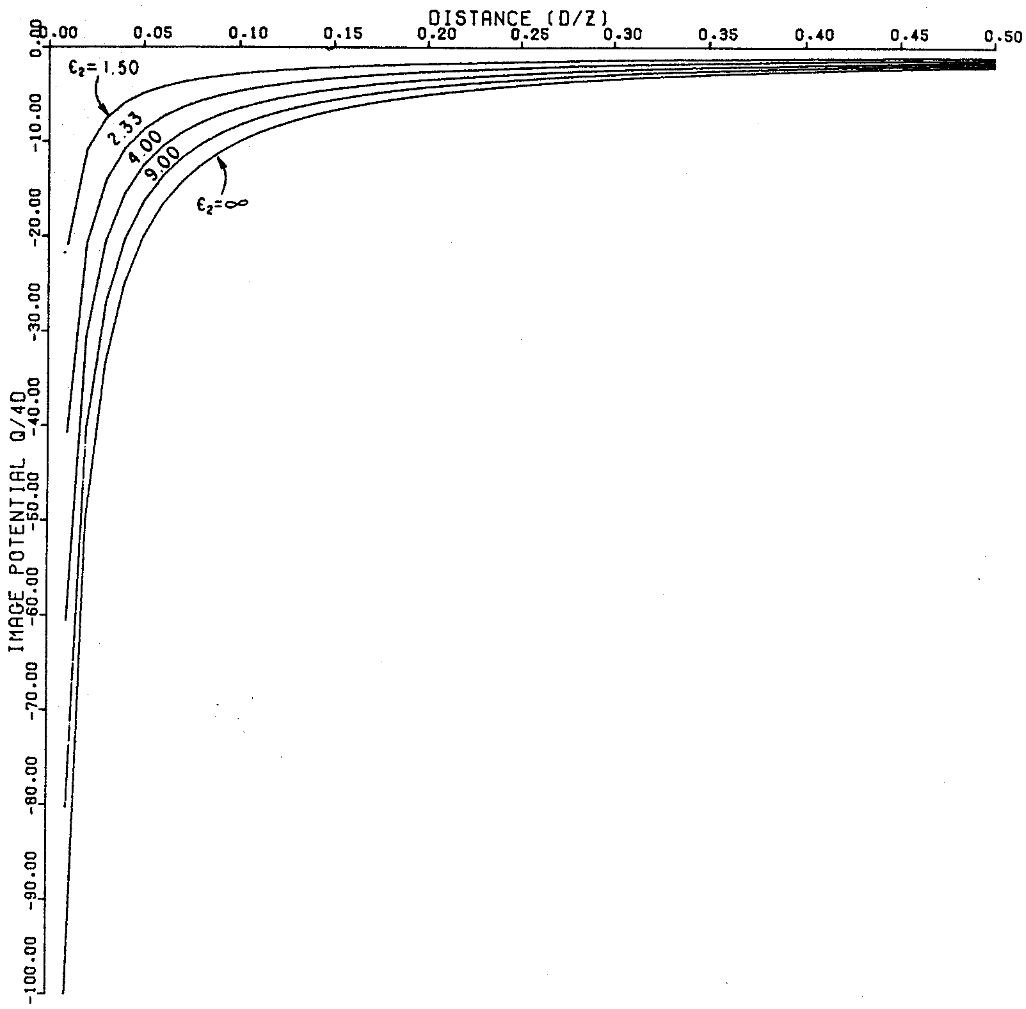


Fig. 4. Image potential in units of $Q/4D$ for a point charge in vacuum in front of a semi-infinite two-dielectric media with a planar interface as a function of $\zeta$ for different values of $\epsilon_2$.

to illustrate some typical cases. Since the integral diverges at $\zeta = \pm 1/2$ we split the integral from 0 to $\infty$ into an integral from 0 to $XM$ plus an integral from $XM$ to $\infty$. The value of $XM$ is chosen high enough such that the integral from $XM$ to $\infty$ can be done analytically, to any degree of accuracy, by approximating the integrand by an exponential. The integral from 0 to $XM$ is then performed through the Gauss' integration method.[16] In Fig. 2 we show a plot of $v_{\rm im}$ as a function of $\zeta$ in the case of a point charge in a dielectric medium between two identical dielectric half-spaces ($L_{12} = L_{32} \equiv \mathcal{L}$), for different values of $\mathcal{L}$. The curve at the bottom correspond to the case when the two half-spaces are conducting ($\mathcal{L} = 1$) and its expression if given by Eq. (45). Figure 3 shows graphs of $v_{\rm im}$ as a function of $\zeta$ in the asymmetric case in which medium 1 is a conductor ($L_{12} = 1$) and media 2 and 3 are dielectrics for different values of $L_{32}$.

In *case II* ($z' > a$) the corresponding expression for the image potential becomes

$$v_{\rm im}(\zeta) = [(QL_{32}/\epsilon_3)/4D][1/(\zeta - 1/2)] - Q\frac{L_{12}}{D}\frac{\epsilon_2}{(\epsilon_2+\epsilon_3)^2}\int_0^\infty d\chi \frac{e^{-(\zeta+1/2)\chi}}{1 - L_{12}L_{32}e^{-\chi}}; \quad 1/2 < \zeta < \infty \qquad (46)$$

where $\zeta \equiv z'/D$, $\chi = 2\kappa D$, and the zero of the potential has been choosen at infinity ($\zeta = \infty$).

In the case of a metallic slab in vacuum ($\epsilon_1 = \epsilon_3 = 1$; $\epsilon_2 = \infty$) Eq. (46) reduces to

$$v_{\rm im}(z') = -Q/4(z' - D/2) \qquad (47)$$

which is the image potential of a point charge in front of an infinite conducting medium.

Performing, as before, a numerical integration of Eq. (46) we show in Fig. 4 a plot of $v_{\rm im}$ as a function of $\zeta$ in the case of metal-dielectric-vacuum system ($L_{12} = 1$; $\epsilon_3 = 1$) for different choices of $\epsilon_2$.

*Case III* ($z' < -a$) is obtained from Eq. (47) by simply changing $\epsilon_1 \leftrightarrow \epsilon_3$ and $\zeta \to -\zeta$.

## V. SYNOPSIS

In this paper we have obtained an exact integral representation for the electrostatic potential, the surface charge density, and the image potential for a point charge in an arbitrary position, in a three-dielectric medium with planar interfaces.

The potential is calculated by the method of images and through a two-dimensional Fourier transform (parallel to the interfaces) we sum the contribution of the infinite array of images which turns out to be a geometrical series. The inverse Fourier transformation gives the corresponding expression in $R$ space in an integral form. Thus this method can be very illustrative for students without the use of complicated mathematics.

The calculation of the other electrostatic quantities follows immediately. For the image potential we perform the integrals numerically and we show plots for different cases.

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